Geometric combinatorics: example sheet 3*

1. Suppose $\mathcal{F}_1 \subset 2^X$ and $\mathcal{F}_2 \subset 2^X$ are two families of finite VC-dimension. Show that the family $\{S_1 \cup S_2 : S_1 \in \mathcal{F}_1,\ S_2 \in \mathcal{F}_2\}$ has finite VC-dimension.

2. Let X be an n-element set. Construct a family $\mathcal{F} \subset 2^X$ of VC-dimension d with $g(d,n) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}$ elements. Is it unique?

3. Let $(X, \mathcal{F})$ be a set family. Recall that an ε-approximation to a finite set $P \subset X$ is a set $P' \subset P$ such that for all $S \in \mathcal{F}$ we have
$$\left| \frac{|P \cap S|}{|P|} - \frac{|P' \cap S|}{|P'|} \right| \leq \varepsilon.$$
Show that if VC-dimension of $\mathcal{F}$ is d, then for every finite $P \subset X$ there is a $(1/r)$-approximation P' of size $|P'| \leq c_d r^2 \log r$.

4. A weak ε-approximation is analogous to (strong) ε-approximations, except that they need not be subsets of sets they approximate (the condition $P' \subset P$ is dropped). Show that there is no function $f(\varepsilon)$ such that for every finite set $P \subset \mathbb{R}^2$ there is a weak ε-approximation $P' \subset \mathbb{R}^2$ to P with respect to the family of convex sets of size $|P'| \leq f(\varepsilon)$.

5. ("Second selection theorem") Suppose that $P \subset \mathbb{R}^d$ is a set of n points, $\mathcal{F}$ is a family of all the $\binom{n}{d+1}$ simplices spanned by P, and $\mathcal{F}' \subset \mathcal{F}$ is a family of size $\alpha\binom{n}{d+1}$. Show that there is a point $p \in \mathbb{R}^d$ that is contained in at least $\beta\binom{n}{d+1}$ members of $\mathcal{F}'$, where $\beta = \beta(\alpha, d) > 0$.

6. (Same-type lemma, special case) The *orientation* of a triangle $p_1p_2p_3$ is either clockwise or counterclockwise depending on the sign of the determinant
$$\det \begin{bmatrix} 1 & p_{11} & p_{12} \\ 1 & p_{21} & p_{22} \\ 1 & p_{31} & p_{32} \end{bmatrix}.$$
Suppose $P_1, P_2, P_3 \subset \mathbb{R}^2$ are three sets of points in the plane, each containing n points. Show that there are subsets $P_1' \subset P_1$, $P_2' \subset P_2$, $P_3' \subset P_3$ of sizes $|P_1'|, |P_2'|, |P_3'| \geq cn$, where $c > 0$ is an absolute constant, so that

*This example sheet is from `http://www.borisbukh.org/GeoCombEaster10/example3.pdf`.

the triangles of the form $p_1p_2p_3$, $p_1 \in P_1$, $p_2 \in P_2$, $p_3 \in P_3$, all have the same orientation. Hint: Cut the sets P_1, P_2, P_3 by appropriately chosen lines.