Geometric combinatorics: example sheet 2*

1. Suppose $P_1 \subset \mathbb{R}^d$ and $P_2 \subset \mathbb{R}^d$ are finite sets that satisfy $\operatorname{conv}(P_1) \cap \operatorname{conv}(P_2) = \emptyset$. Convince your non-mathematical friend that there is a hyperplane h such that P_1 and P_2 are on different sides of h. *Next, convince yourself that the same holds if P_1 and P_2 are arbitrary compact sets.

2. Suppose the point $p \in \mathbb{R}^d$ is at depth αn with respect to the n-point set $P \subset \mathbb{R}^d$, and $\alpha > 0$. Show that there is a constant $\beta = \beta(\alpha, d) > 0$ such that the point p is in at least $\beta\binom{n}{d+1}$ of the simplices spanned by P.

3. Parametrize the directions in $\mathbb{R}^d$ by the unit sphere $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. Let "almost all" mean "for all except possibly a set of measure zero". Let $C \subset \mathbb{R}^d$ be a compact convex set. Suppose $d = 2$, and show that for almost all directions $v \in \mathbb{S}^{d-1}$ there is a unique point $p \in C$ attaining $\min\langle v, p\rangle$. *(Very difficult) What about $d \geq 3$?

4. Assuming the solution to the previous exercise, give a proof of the fractional Helly theorem for $d = 2$ that uses $\min\langle v, p\rangle$ for appropriate v in place of lexicographic minima.

5. Let $P \subset \mathbb{R}^2$ be a finite set of points, not all of which lie on a single line. Show that there is a line that passes through exactly two of them.

6. (Coloured Helly theorem) Let $\mathcal{C}_1, \dots, \mathcal{C}_{d+1}$ be finite families of convex sets in $\mathbb{R}^d$ such that for every

$$C_1 \in \mathcal{C}_1, \dots, C_{d+1} \in \mathcal{C}_{d+1}$$

   the intersection $\bigcap C_i$ is non-empty. Then for some i, we have $\bigcap \mathcal{C}_i \neq \emptyset$. Hint: consider lexicographic minima.

7. A *convexity space* is a ground set X, together with a collection $\mathcal{C}$ of subsets of X, called *convex sets*, with condition that the empty set and X itself are convex, and $\mathcal{C}$ is closed under arbitrary intersections. A *convex hull* of $S \subset X$ is the smallest convex set containing S, $\operatorname{conv} S = \bigcap\{C \in \mathcal{C} : S \subset C\}$. For example, $\mathbb{R}^d$ together with the family of standard convex sets is a convexity space. For a convexity space $(X, \mathcal{C})$ let N_r be the least number N such that

*This example sheet is from `http://www.borisbukh.org/GeoCombEaster10/example2.pdf`.

every set $P \subset X$ of size $|P| = N$ can be partitioned into r sets $P = P_1 \cup \cdots \cup P_r$ so that $\operatorname{conv}(P_1) \cap \cdots \cap \operatorname{conv}(P_r) \neq \emptyset$. Show that if N_2 is finite, then so is N_r for every r. *(Open problem) Show that $N_r \leq (N_2 - 1)(r - 1) + 1$, which in particular would mean that Radon's theorem implies Tverberg's theorem.