Geometric combinatorics: example sheet 1*

1. Recall that a convex set is a set S such that $\operatorname{conv} S = S$. Show that $S \subset \mathbb{R}^d$ is convex if and only if for every two points $x, y \in S$, the set $\operatorname{conv}(\{x, y\})$ is contained in S.

2. Show that if S is compact closed set, then so is $\operatorname{conv} S$. Suppose $\mathcal{F}$ is an infinite family of compact convex sets in $\mathbb{R}^d$, and every collection of $d+1$ of them has a non-empty intersection. Show that all of them have a non-empty intersection.

3. (Alternative proof of centrepoint theorem) For a convex set $C \subset \mathbb{R}^2$, let $x(C)$ be the x-coordinate of the leftmost point of C. Let $P \subset \mathbb{R}^2$ be an n-point point set in general position. Among all the pairs (H_1, H_2) of halfspaces containing more than $\frac{2}{3}n$ points of P, consider the pair for which $x(H_1 \cap H_2)$ is maximal. Show that the leftmost point of $H_1 \cap H_2$ is at depth at least $\frac{1}{3}n$. *Generalize to $\mathbb{R}^d$, any d.

4. Let $P_1, P_2 \subset \mathbb{R}^d$ be two finite sets whose convex hulls meet. Show that there are subsets $P_1' \subset P_1$ and $P_2' \subset P_2$ of total size $|P_1'| + |P_2'| \leq d + 2$ such that $\operatorname{conv} P_1 \cap \operatorname{conv} P_2 \neq \emptyset$. *Suppose the convex hulls of P_1, P_2, P_3 intersect. Then there are three subsets $P_1' \subset P_1$, $P_2' \subset P_2$ and $P_3' \subset P_3$ of total size $|P_1'| + |P_2'| + |P_3'| \leq f_3(d)$ whose convex hulls also intersect. How small can you make $f_3(d)$?

5. Let $f \colon \mathbb{R}^d \to [0, \infty)$ be a continuous function with integral $\int f = 1$. A point $p \in \mathbb{R}^d$ is said to be at depth α with respect to f if for every closed halfspace containing p we have $\int_H f \geq \alpha$. Show that for every such f there is a point at depth at least $1/(d+1)$. *Extend this result to Borel probability measures on $\mathbb{R}^d$.

6. *Let $V \subset \mathbb{R}^d$ be a finite set of vectors, each of norm at most 1, with $\sum_{v \in V} v = 0$. Show that there is an ordering $v_1, \ldots, v_n$ of these vectors such that for all $k = 1, \ldots, n$ the norm of

$$v_1 + \cdots + v_k$$

*This example sheet is from `http://www.borisbukh.org/GeoCombEaster10/example1.pdf`.

is at most d. Hint: Construct a nested sequence of sets $V_{d+1} \subset \cdots \subset V_n = V$ where $|V_k| = k$, together with functions $\alpha_k \colon V_k \to [0,1]$ satisfying

$$\sum_{v \in V_k} \alpha_k(v) v = 0$$
$$\sum_{v \in V_k} \alpha_k(v) = k - d.$$