Discrete Math: homework #6*
Due 17 November 2021, at 9:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `andrewid_discr_hwnum.tex` and `andrewid_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

You can earn 1 points of extra credit by submitting exactly 2 problems before 9am on 12 November 2021. Resubmissions void the extra credit.

1. [2] Prove that there is a constant $c_t > 0$ such that in every 2-coloring of K_n there are at least $c_t n^t + o(n^t)$ monochromatic K_t's.

2. [2] Prove that there is a constant $c_t > 0$ such that in every n-vertex graph with m edges, where $c_t m \geq n^{2-1/t}$, there are at least $n^{2t}(m/n^2)^{t^2}$ copies of $K_{t,t}$.

3. [2] Prove that for every $\varepsilon > 0$ there exists n_0 such that whenever p is a prime satisfying $p \geq n_0$ and $\lambda \in [p-2]$, then every set $A \subseteq \mathbb{Z}/p\mathbb{Z}$ of size $|A| \geq \varepsilon p$ contains a solution to the equation $x + \lambda y = (\lambda+1)z$ with x, y, z not being all equal.

4. [2] For a continuous function $f\colon \mathbb{R} \to [0,1]$ and a bounded interval $I \subset \mathbb{R}$, we define $N(I) = \int_I f(x)\,dx$ and $d(I) = N(I)/\operatorname{len}(I)$, where $\operatorname{len}(I)$ is the length of I. The interval I is called *ε-regular* if for every subinterval I' of length at least $\varepsilon \cdot \operatorname{len}(I')$ we have $|d(I) - d(I')| \leq \varepsilon$.

   Show that for every $\varepsilon > 0$ there exists a number $M(\varepsilon)$ with the following property: For every continuous function $f\colon \mathbb{R} \to [0,1]$ there exists a partition of the interval $[0,1]$ into intervals $I_1, \ldots, I_r$ with $r \leq M(\varepsilon)$ such that

$$\sum_{I_t \text{ is } \varepsilon\text{-regular}} \operatorname{len}(I_t) \geq 1 - \varepsilon.$$

*This homework is from `http://www.borisbukh.org/DiscreteMath21/hw6.pdf`.

5. [2] Let G be the bipartite graph with parts A, B, each of which is a copy of $[n]$, and in which $a \in A$ is joined to $b \in B$ if $a \leq b$. Prove that, for sufficiently large n, the graph G does not admit an $(1/100)$-regular partition $J \cup V_1 \cup \cdots \cup V_\ell$ with $\ell = O(1)$, in which *all* pairs (V_i, V_j) are regular.