Discrete Math: homework #5*
Due 3 November 2021, at 9:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `andrewid_discr_hwnum.tex` and `andrewid_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

You can earn 1 points of extra credit by submitting exactly 2 problems before 9am on 27 October 2021. Resubmissions void the extra credit.

1. [2] Show that for each r there is $m = m(r)$ with the following property: If $\mathcal{H} = (V, E)$ is an r-uniform hypergraph such that every vertex is in at least m edges, then it is possible to find a partition $E = E_1 \cup E_2$ such that for each vertex v there are $e_1 \in E_1$ and $e_2 \in E_2$ such that v is both in e_1 and e_2.

2. [1+1] Let G be a graph, and let H be a graph obtained from G by deleting each edge with probability $1/2$. Let $\chi(G)$ and $\chi(H)$ be the chromatic numbers of G and H.

   (a) Show that $\mathbb{E}[\chi(H)] \geq \chi(G)^{1/2}$. (Hint: consider the complement of H).

   (b) Show that $\Pr[\chi(H) < c\chi(G)^{1/2}] \leq f(c)$ for an explicit function f satisfying $f(c) \to 0$ as $c \to 0$.

3. [2] A path of even length $P = v_1 v_2 \cdots v_{2k}$ in a graph (V, E) with a vertex coloring $f\colon V \to [r]$ is *periodic* if $f(v_j) = f(v_{j+k})$ for all j satisfying $1 \leq j \leq k$. Prove that there exists a constant r such that every graph G with maximum degree 5 admits a vertex r-coloring in which no path of of any even length is periodic.

4. [2] Show that there exists a function $t(n)$ satisfying $t(n) = o(\sqrt{n})$ such that for each n there exists an interval I_n of length $t(n)$ satisfying the following: If G is a graph chosen uniformly at random among all graphs on the vertex set $[n]$, then $\Pr\bigl[\chi(G) \in I_n\bigr] \geq 0.99$. (Note that we proved a weaker result in class.)

*This homework is from `http://www.borisbukh.org/DiscreteMath21/hw5.pdf`.