Discrete Math: homework #3*
Due 6 October 2021, at 9:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `andrewid_discr_hwnum.tex` and `andrewid_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

You can earn 1 points of extra credit by submitting exactly 3 problems before 9am on 29 September 2021. Resubmissions void the extra credit.

1. (a) [2] Show that for each n and r there exists N with the following property: For every coloring $\chi\colon \binom{[N]}{2} \times \binom{[N]}{2} \to [r]$ of ordered pairs of edges of K_N, there are sets $X, Y \subset [N]$ of size $|X| = |Y| = n$ such that χ is monochromatic on $\binom{X}{2} \times \binom{Y}{2}$.

   (b) [1] Prove or disprove that for every coloring $\chi\colon \binom{\mathbb{N}}{2} \times \binom{\mathbb{N}}{2} \to [r]$ there exist infinite sets $X, Y \subset \mathbb{N}$ such that χ is monochromatic on $\binom{X}{2} \times \binom{Y}{2}$.

2. [1] Prove that for each coloring $\chi\colon \binom{\mathbb{N}}{2} \to \mathbb{N}$ there exists an infinite subset $X \subset \mathbb{N}$ such that either

   (a) χ is monochromatic on $\binom{X}{2}$, or

   (b) no two edges in $\binom{X}{2}$ receive the same color, or

   (c) $\chi(\{i, j\})$ depends only on $\min(i, j)$, or

   (d) $\chi(\{i, j\})$ depends only on $\max(i, j)$.

3. (a) [1] Show that for each $\alpha > 0$ there exists $\beta > 0$ and n_0 such that whenever $n \geq n_0$ and $A \subset [n]$ is of density at least α, the number of 3-APs in A is at least βn^2.

   (b) [1] A *d-Roth-cube* is a set of the form $x_0 + x_1 \cdot \{0, 1, 2\} + \cdots + x_d \cdot \{0, 1, 2\}$ where $x_1, \ldots, x_d$ are all non-zero. Deduce from part (a) that for every $\alpha > 0$

*This homework is from `http://www.borisbukh.org/DiscreteMath21/hw3.pdf`.

and $d \in \mathbb{N}$ there exists $n = n(\alpha, d)$ such that every set $A \subset [n]$ of density at least α contains a d-Roth-cube. (You may do this even if you did not solve part (a).)

4. Let $s, t \in \mathbb{N}$. The vertex set of a graph G is a disjoint union of infinitely many blocks, each block being a set of size t. Inside any set of s distinct blocks there is an edge that goes between two different blocks. Show that in G there is an infinite path visiting no block more than once.

5. [2] Let $R_3(k_1, k_2)$ denote the 3-uniform Ramsey numbers for two colors. Show that $R_3(k, 4) = 2^{\Omega(k)}$. [Hint: consider a random orientation of the edges of a complete graph. Use it to define a coloring of the 3-uniform hypergraph on the same vertex set.]

6. (a) [2] Show that for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ with the following property. Whenever α is a real number there are integers q and p such that $1 \le q \le N$ and
$$|q^2\alpha - p| \le \varepsilon.$$
(Hint: van der Waerden's theorem.)

   (b) (Open problem; much extra credit) May one take $N = \varepsilon^{-1-o(1)}$ in the above?

7. [2+(1 extra credit)] The *step* of an arithmetic progression $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$ is defined to be $|d|$, the Euclidean norm of d.

   (a) Show that there is an r such that for each n there is a coloring $\chi\colon \mathbb{R}^n \to [r]$ that contains no monochromatic 3-AP with step 1. [Hint: choose $\chi(x)$ that depends only on $|x|$.]

   (b) [Extra credit] Show that for every r there is an n such that for every r-coloring χ of $\mathbb{R}^n$ there is a 4-AP $\{x_1, x_2, x_3, x_4\}$ with step 1 that satisfies $\chi(x_1) = \chi(x_4)$ and $\chi(x_2) = \chi(x_3)$. Here it is understood that x_1, x_2, x_3, x_4 are in order, i.e., they satisfy $2x_2 = x_1 + x_3$ and $2x_3 = x_2 + x_4$. [Hint: Consider the coloring of $\{0, \lambda, 2\lambda, 3\lambda\}^n \subset \mathbb{R}^n$, for a suitable λ.]