# Discrete Math: homework #1*
# Due 8 September 2021, at 9:00am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `andrewid_discr_hwnum.tex` and `andrewid_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. [2] Let $k, l, m$ be arbitrary natural numbers. Let $ES(k, l, m)$ be the length of the longest sequence of real numbers that contains *neither*

   - a strictly increasing subsequence of length $k$, *nor*
   - a strictly decreasing subsequence of length $l$, *nor*
   - a constant subsequence of length $m$.

   Find $ES(k, l, m)$.

2. [1+1]

   (a) Let $r$ and $b$ be some fixed natural numbers. Edges of a complete graph on $[n] = \{1, 2, \ldots, n\}$ are colored red and blue. Show that if $n$ is large enough, then the graph contains either an $r$-vertex red clique or a blue path of length $b$ whose vertices are in increasing order.

   (b) Find the smallest $n$ such that (a) holds.

3. (a) [2] Show that every sequence of distinct real numbers either contains an increasing subsequence of length $s+1$ or can be partitioned into at most $s$ decreasing subsequences.

   (b) [1] Deduce Erdős–Szekeres theorem on monotone subsequence from the statement in part (a). (You may do this part without doing part (a)).

*This homework is from `http://www.borisbukh.org/DiscreteMath21/hw1.pdf`.

4. For a function $\phi\colon \mathbb{R} \to \mathbb{R}$ and a $k$-tuple $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ we define $\phi(x) = \big(\phi(x_1), \ldots, \phi(x_k)\big)$.

   Consider sequences with entries in $\mathbb{R}^k$. Call two such sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ *isomorphic* if there is an order-preserving bijection $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $\phi(a_i) = b_i$ for all $i$. Call a sequence $a_1, \ldots, a_n$ *homogeneous* if there is an arbitrarily long sequences all of whose $n$-element subsequences are isomorphic to $a_1, \ldots, a_n$

   (a) [0, do not have to turn in] Find all homogeneous sequences for the case $k = 1$.

   (b) [1 extra credit] Show that for each $k$, there is $f(k)$ (which depends only on $k$) such that the number of isomorphism classes of homogeneous sequences of length $n$ is at most $f(k)$.