21-701 4 Oct 2021

Distinct sums

$f(n) = \max\{k : \exists A \in \binom{[n]}{k}$ with distinct sums$\}$

$2^{f(n)} \le kn \qquad [0, kn]$

$\Rightarrow f(n) \le \log_2 n + \log_2 \log_2 n + O(1)$

2^k sums

Define: $X = \sum X_a$ for $a \in A$

$$X_a = \begin{cases} a & \text{with prob } 1/2 \\ 0 & \text{with prob } 1/2 \end{cases}$$

$$\mathbb{E}[X] = \frac{1}{2}\sum_{a\in A} a$$

$$\mathrm{Var}[X] = \sum_{a\in A} \mathrm{Var}[X_a]$$

$$= \sum_{a\in A} \left(\mathbb{E}[X_a^2] - \mathbb{E}[X_a]^2\right)$$

$$= \sum_a \left(\frac{1}{2}a^2 - \left(\frac{1}{2}a\right)^2\right) = \frac{1}{4}\sum_a a^2$$

$$\le \frac{1}{4}kn^2$$

Chebyshev $\Longrightarrow$

$$\mathbb{P}\left[\,|X - \mathbb{E}[X]| > \lambda\,\right] < \frac{\mathrm{Var}[X]}{\lambda^2} \le \frac{1/4\, kn^2}{\lambda^2} \qquad \lambda = 10\sqrt{k}\,n$$

$$\mathbb{P}\left[X \in [\mu - 10\sqrt{k}n, \mu + 10\sqrt{k}n]\right]$$

$$\ge 1 - \frac{1}{4\cdot 10^2} > \frac{1}{2}$$

By pigeonhole, $\exists m$

$$\mathbb{P}[X = m] \ge \frac{1}{2\cdot 20\sqrt{k}n}$$

$$2^k \le 40\sqrt{k}\,n$$

$$k \le \log_2 n + \frac{1}{2}\log_2 k + O(1)$$

$\Rightarrow \quad f(n) \le \log_2 n + \frac{1}{2}\log_2\log_2 n + O(1)$

$\mathbb{E}[X^k]$

$$\mathbb{P}[|X - \mathbb{E}[X]| \ge \lambda] \le \frac{c_k}{\lambda^k}$$

Instead, $\mathbb{E}[e^{tX}]$ $\quad t>0$

Chernoff's inequality:

Let $X_1, X_2, \ldots, X_n$ independent

$|X_i| \leq 1$ for each i

$X = \sum_{i=1}^{n} X_i$ $\quad \mathbb{E}[X_i] = 0$

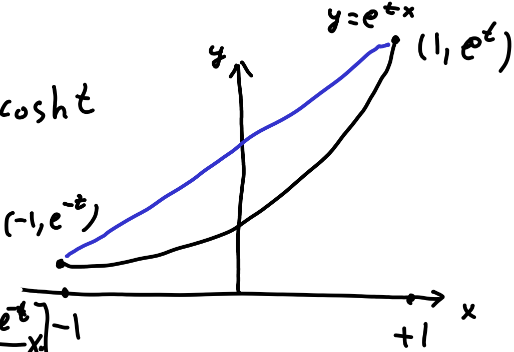
Then $\mathbb{P}[X \geq \lambda] \leq e^{-\lambda^2/2n}$.

Pf: $\mathbb{E}[e^{tX}] = \mathbb{E}[e^{tX_1} \cdot e^{tX_2} \cdot \ldots \cdot e^{tX_n}]$

$= \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}]$ independence

Lemma:

$$\mathbb{E}[e^{tX_i}] \le \frac{e^t + e^{-t}}{2} =: \cosh t$$



Pf:

$$e^{tx} \le \frac{e^t + e^{-t}}{2} + \frac{e^t - e^{-t}}{2} x$$

$$\mathbb{E}[e^{tX_i}] \le \mathbb{E}\left[\frac{e^t + e^{-t}}{2}\right] + \mathbb{E}\left[\frac{e^t - e^{-t}}{2} X_i\right]$$

$$= \frac{e^t + e^{-t}}{2} + 0. \quad \square$$

Note: $\frac{e^t + e^{-t}}{2} \le e^{t^2/2} = \sum_{n \ge 0} (t^2)^n / n! = \sum_{n \ge 0} t^{2n}/n!$

Pf: $\frac{1}{2}(1 + t + t^2/2! + t^3/3! + \dots) + \frac{1}{2}(1 - t + t^2/2! - t^3/3! + \dots) = \sum_n t^{2n}/(2n)!$ $\square$

$$\mathbb{E}[e^{t\bar{X}}] = \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] \le \prod_{i=1}^{n} e^{t^2/2} = e^{t^2 n/2}$$

$$\mathbb{P}[X \ge \lambda] = \mathbb{P}[e^{t\bar{X}} \ge e^{t\lambda}] \overset{\text{Markov}}{\le} \frac{\mathbb{E}[e^{t\bar{X}}]}{e^{t\lambda}}$$

$$= e^{t^2 n/2 - t\lambda} \qquad t = \lambda/n$$

$$= e^{-\lambda^2/2n} \qquad \blacksquare$$

$$\mathbb{P}[\bar{X} \le -\lambda] = \mathbb{P}[(-\bar{X}) \ge \lambda] \le e^{-\lambda^2/2n}$$

$$\mathbb{P}[|X| \ge \lambda] \le 2e^{-\lambda^2/2n}$$

Binomial random variable

$$X = X_1 + \dots + X_n \quad \text{independent}$$

$$X_i = \begin{cases} 1 & \text{with prob } 1/2 \\ 0 & \text{with prob } 1/2 \end{cases}$$

Apply Chernoff to

$$Y = 2\left(X - \frac{n}{2}\right)$$

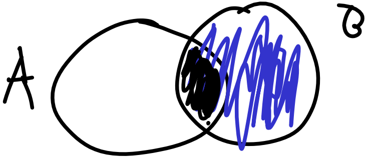
$$\boxed{\mathbb{P}\left[\left|X - \frac{n}{2}\right| \geq \lambda\right] \leq e^{-2\lambda^2/n}.}$$

Conditional probability

Events A, B

$$\mathbb{P}[A \mid B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

(undefined if $\mathbb{P}[B]=0$)



Conditional expectation

random variable X, event B

$$\mathbb{E}[X \mid B] = \sum_x x \, \mathbb{P}[X=x \mid B]$$

Observation:

If $B_1, \dots, B_n$ are events s.t. $B_i \cap B_j = \emptyset$ and $\bigcup B_i = \Omega$

($\Leftarrow \mathbb{P}[\bigcup B_i] = 1$)

then
$$\mathbb{P}[A] = \sum_{i=1}^{n} \mathbb{P}[A \mid B_i] \, \mathbb{P}[B_i]$$
$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid B_i] \, \mathbb{P}[B_i]$$