Walk through Combinatorics: homework #8*
Due 5 December 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

You can earn 0.5 points of extra credit by submitting exactly 2 problems before the start of class on 28 November 2018. Resubmissions void the extra credit. (You may resubmit part (c) of problem 2 without voiding extra credit.)

1. [2+2+1] Let n be even.

   (a) Let A be a random $n/2$-element set in $[n]$ chosen uniformly. Show that
   $$\Pr\Big[\big||A \cap [n/2]| - n/4\big| \geq n/8\Big] \leq 2\exp(-n/5000)$$

   (b) Let A be a random $n/2$-element set in $[n]$ chosen uniformly. Show that
   $$\Pr\Big[|A \cap [n/2]| \leq n/1000\Big] \leq 2^{-0.9n+o(n)}.$$

   (c) Let $U = \{u_1, \dots, u_n\}$ and $V = \{v_1, \dots, v_n\}$ be two disjoint sets. Define a random bipartite graph G with parts U and V as follows: Pick three random permutations π_1, π_2, π_3 of $[n]$, and connect vertex v_i to each of $u_{\pi_1(i)}, u_{\pi_2(i)}, u_{\pi_3(i)}$ by an edge. We allow parallel edges if some $\pi_1(i), \pi_2(i), \pi_3(i)$ are equal. Call a pair of sets $U' \subset U$ and $V' \subset V$ *bad* if $|U'| = |V'| = n/2$ and there are fewer than $n/1000$ edges between U' and V'. (In case of parallel edges, we count edges with multiplicity.)

   Prove that
   $$\Pr[\exists \mathrm{bad}(U', V')] = o(1) \qquad \text{as } n \to \infty.$$

   (You may use the preceding parts even if you did not solve them.)

*This homework is from `http://www.borisbukh.org/DiscreteMath18/hw8.pdf`.

2. [1+1+(1 bonus)] Let G be a graph, and let H be a graph obtained from G by deleting each edge with probability $1/2$. Let $\chi(G)$ and $\chi(H)$ be the chromatic numbers of G and H.

   (a) Show that $\mathbb{E}[\chi(H)] \geq \chi(G)^{1/2}$. (Hint: consider the complement of H).

   (b) Show that $\Pr[\chi(H) < c\chi(G)^{1/2}] \leq f(c)$ for an explicit function f satisfying $f(c) \to 0$ as $c \to 0$.

   (c) Let H' be a graph obtained from G by deleting each edge with probability $2/3$. Show that $\Pr[\chi(H') < c\chi(G)^{1/3}] \leq f'(c)$ for an explicit function f' satisfying $f(c) \to 0$ as $c \to 0$.

3. [2] For a permutation π, let $X = X(\pi)$ be the least number m such that π is a product of m cycles. Show that there is a constant C such that if one picks π uniformly at random from the symmetric group S_n, then

$$\Pr\big[|X - \mathbb{E}[X]| \geq C\sqrt{n}\big] \leq 0.01$$