Walk through Combinatorics: homework #7*
Due 14 November 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

You can earn 0.5 points of extra credit by submitting exactly 2 problems before the start of class on 7 November 2018. Resubmissions void the extra credit.

1. [Bonus problem] Vote!

2. [2] Show that that there exists a constant c with the following property. Whenever G is a graph on n vertices and $\frac{1}{2}\binom{n}{2}$ edges, then there is a subset U of vertices such that $|e(U) - \frac{1}{2}\binom{|U|}{2}| \geq cn^{3/2}$.

3. [2] Prove that there is an absolute constant $c > 0$ with the following property. Let A be an n-by-n matrix with pairwise distinct real entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

4. [2+(1 extra credit)]

   (a) Let $\mathcal{S}$ be a set of binary words of finite length and assume that no word in $\mathcal{S}$ is a prefix of another. Let N_ℓ be the number of words of length ℓ in $\mathcal{S}$. Prove that $\sum_\ell N_\ell 2^{-\ell} \leq 1$.

   (b) Let $w_1 w_2$ denote the concatenation of words w_1 and w_2. Suppose that $\mathcal{S}$ is a set of binary words of finite length such that for every word w there is at most one way of writing it as $w = w_1 \dots w_k$ for some $k \in \mathbb{Z}_+$ and some $w_i \in \mathcal{S}$. Prove that the inequality $\sum_\ell N_\ell 2^{-\ell} \leq 1$ continues to hold.

*This homework is from `http://www.borisbukh.org/DiscreteMath18/hw7.pdf`.

5. [1] Let A be a random subset of $[n]$ such that $\Pr[k \in A] = p$ and the events $k \in A$ are independent. Let E denote the event that A contains a 4-term arithmetic progression.

   Find a function $t(n)$ such that

   - if $p = o(t(n))$, then $\Pr[E] \to 0$; and
   - if $p = \omega(t(n))$, then $\Pr[E] \to 1$.

6. [2] Write down an explicit positive constant c such that the following holds for all natural numbers n. Suppose $a_1, \ldots, a_n$ are n real numbers satisfying $\sum_{i=1}^n a_i^2 = 1$, and $\epsilon_1, \ldots, \epsilon_n \in \{-1, +1\}$ are uniformly and independently chosen signs. Then

$$\Pr\left[\left|\sum_{i=1}^{n} \epsilon_i a_i\right| \leq 1\right] \geq c.$$