Walk through Combinatorics: homework #5*
Due 17 October 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

You can earn 0.5 points of extra credit by submitting exactly 3 problems before the start of class on 12 October 2018. Resubmissions void the extra credit.

1. [1] Show that $n! \geq (n/e)^n$ by estimating $[z^n]e^z$ via Cauchy's inequality for the complex integral.

2. [2+(1 extra credit)]

   (a) Let $m \in \mathbb{N}$ be arbitrary fixed number. Let S_n be the number of sequences of $+1$ and -1 of length n such that all the partial sums lie in the intervals $[-m, m]$ and the total sum is zero. Find a closed form for the generating function $\sum S_n z^n$. [A "closed form" here means an expression in z and m that involves only operations of addition, multiplication, exponentiation and their inverses.]

   (b) Show that $S_n = (c_m + o(1))^n$ and determine an asymptotics for $|2 - c_m|$ as $m \to \infty$.

3. [2] Let $A \subset \mathbb{N}$ be an infinite set of integers. Let R_n be the number of ways to write n as $n = a_1 + a_2$ with $a_1, a_2 \in A$, where the order of summands does not matter (i.e. $5 = 3 + 2$ and $5 = 2 + 3$ are the same). Show that $R_n = C$ cannot hold for all $n \geq n_0$, for any constants C and n_0. (Hint: singularities of the generating function.)

4. [2+1]

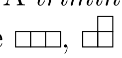
*This homework is from `http://www.borisbukh.org/DiscreteMath18/hw5.pdf`.

(a) An *ordering* of $x_1, \ldots, x_n$ is a permutation of $x_1, \ldots, x_n$ with relations $<$ and $=$ inserted between them. Two orderings are considered the same if they differ only in the order elements that are joined by equal signs. For example, $x_3 < x_1 = x_2$ is same as $x_3 < x_2 = x_1$.

Let C_n be the number of orderings for $x_1, \ldots, x_n$. Find the exponential generating function for the sequence (C_n).

(b) Find an asymptotics for C_n. (It should be of the form $f(n)(1 + o(1))$ for explicit $f(n)$ that involves only addition, multiplication, exponential, trigonometric, factorial functions and their inverses.)

5. [2] Let X_n be the number of alternating permutations on $[n]$ with $\pi(1) = 1$. Let Y_n be the number of alternating permutations on $[n]$ with $\pi(1) = 2$. Is it true that $X_n \geq Y_n$ for all sufficiently large n? Justify.

6. [2] A *trimino* consists of three connected unit squares. In other words, the triminos are  and whatever can be obtained from these by rotation and reflection. Let T_n be the number of ways to cut 3-by-n rectangle into triminos. Find a closed form for $f(z) = \sum T_n z^n$.

LaTeX tip: The in-line pictures above were produced by TikZ package and code

```
\tikz[scale=0.2]{
      \draw (0,0) rectangle (1,1);
      \draw (1,0) rectangle (2,1);
      \draw (2,0) rectangle (3,1);}
```

and

```
\tikz[scale=0.2]{
      \draw (0,0) rectangle (1,1);
      \draw (1,0) rectangle (2,1);
      \draw (1,1) rectangle (2,2);}
```

respectively.