Walk through Combinatorics: homework #3*
Due 28 September 2018, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. [2] Let $d \in \mathbb{N}$ and $\alpha > 0$ be fixed. Show that there are at least $\Omega(n^{d+1})$ of Hilbert d-cubes in any set $A \subset [n]$ of size $|A| \geq \alpha n$.

2. A set S is *rainbow* with respect to a coloring if every element of S receives a different color.

   (a) [2] Show that for each k there is a $c = c(k) > 0$ such that if $[n]$ is colored in any number of colors, and no color occurs more than cn times, then there is a rainbow arithmetic progression of length k.

   (b) [1] Deduce, using Szemerédi's theorem, that for each k there an n such that if $[n]$ is colored (in any number of colors), then there is a non-trivial arithmetic progression of length k that is either monochromatic or rainbow.

3. [2] Let $\alpha, \beta > 0$ be arbitrary real numbers. Prove that there exists an N such that every $A \subset [N]$ of density at least α contains three numbers of the form $a, a + \lfloor \beta t \rfloor, a + \lfloor 2\beta t \rfloor$ for some integers a and $t > 0$

4. [2] Let $r \in \mathbb{N}$ be a fixed number. Let $\chi\colon \mathbb{N} \to [r]$ be a coloring. Show that, with at most $o(N)$ exceptions, all the even integers $m \in [N]$ can be written in the form $m = x + y$ with distinct x, y of the same color. [Hint: Szemerédi's cube lemma]

*This homework is from `http://www.borisbukh.org/DiscreteMath18/hw3.pdf`.