Walk through Combinatorics: homework #7*
Due 20 November 2017, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

You can earn 0.5 points of extra credit by submitting exactly 2 problems before the start of class on 13 November 2017. Resubmissions void the extra credit.

1. [2+1] Let $\mathcal{F} = \{(A_1, B_1), \ldots, (A_k, B_k)\}$ be a family of pairs of subsets of $[n]$ such that
$$A_i \cap B_j = \emptyset \iff i = j.$$
   (a) Prove that
$$\sum_{i=1}^{k} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$
   (b) Give infinitely many examples of families $\mathcal{F}$ for which the inequality is sharp (i.e. the sum on the left-hand side is equal to 1).

2. [2] Show that that there exists a constant c with the following property. Whenever G is a graph on n vertices and $\frac{1}{2}\binom{n}{2}$ edges, then there is a subset U of vertices such that $|e(U) - \frac{1}{2}\binom{|U|}{2}| \geq cn^{3/2}$.

3. [2] Prove that there is an absolute constant $c > 0$ with the following property. Let A be an n-by-n matrix with pairwise distinct real entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

4. [2+(1 extra credit)]

*This homework is from `http://www.borisbukh.org/DiscreteMath17/hw7.pdf`.

(a) Let $\mathcal{S}$ be a set of binary words of finite length and assume that no word in $\mathcal{S}$ is a prefix of another. Let N_ℓ be the number of words of length ℓ in $\mathcal{S}$. Prove that $\sum_\ell N_\ell 2^{-\ell} \leq 1$.

(b) Let $w_1 w_2$ denote the concatenation of words w_1 and w_2. Suppose that $\mathcal{S}$ is a set of binary words of finite length such that for every word w there is at most one way of writing it as $w = w_1 \dots w_k$ for some $k \in \mathbb{Z}_+$ and some $w_i \in \mathcal{S}$. Prove that the inequality $\sum_\ell N_\ell 2^{-\ell} \leq 1$ continues to hold.

5. [1+1] In both problems below, A is a set of n positive *real* numbers.

(a) Suppose $\vec{t} = (t_1, \dots, t_m) \in \mathbb{Z}^m$ is an integer vector such that $t_1 + \cdots + t_m \neq 0$. Call a set B $\vec{t}$*-free* if there are no $b_1, \cdots, b_m \in B$ satisfying

$$t_1 b_1 + \cdots + t_m b_m = 0.$$

Show that there is a constant $c > 0$ (depending only on t's) so that A contains a $\vec{t}$-free set B of size at least cn.

(b) Show that A contains a $(1, 1, -1)$-free subset of *strictly* more than $n/3$ elements.