Walk through Combinatorics: homework #8*
Due never

Homework must be NOT be submitted. It is for your own practice. Feel free to discuss it with me in office hours though.

1. (a) Let A be a finite non-empty subset of an abelian group G. Show that if $|A+A|=|A|$, then A is a coset of a subgroup of G.
   (b) What happens if G is not abelian?

2. Recall that $\lambda \cdot A = \{\lambda a : a \in A\}$. Suppose $|A + 2\cdot A| \le K|A|$. Show that $|A + 2^t \cdot A| \le K^t|A|$ for all $t \in \mathbb{N}$.

3. (a) Let C be a set of m unit circles in $\mathbb{R}^2$ and let P be a set of n points in $\mathbb{R}^2$. Let $I(P,C)$ be the number of pair $(p,c) \in P \times C$ such that $p \in c$. Prove that $I(P,C) = O\big((mn)^{2/3} + m + n\big)$.
   (b) Show that if $P \subset \mathbb{R}^2$ is of size $|P| = n$, then there are $O(n^{4/3})$ pairs of points in P that are distance 1 apart.
   (c) (Open problem; much extra credit) Show that in (b) one can replace $O(n^{4/3})$ by $o(n^{4/3})$, probably by $O(n^{1+\varepsilon})$ for every $\varepsilon > 0$.

4. (a) Suppose $A, B_1, \dots, B_m$ are non-empty finite subsets of an abelian group G, and $|A + B_i| \le K|A|$ for every $i = 1, 2, \dots, m$. Show that $|B_1 + B_2 + \dots + B_k| \le K^m|A|$. (Hint: consider Cartesian products $A' = A^n$ and $B_i' = B_i^n$ inside G^n, and then look at $|A' + B_i'|$.)
   (b) Suppose $|A + B_i| \le K_i|A|$ for each $i = 1, 2, \dots, m$. Show the inequality $|B_1 + \dots + B_m| \le K_1K_2 \dots K_m|A|$.

5. Let $A \subset \mathbb{Z}$ be finite and non-empty. Put $K = |A+A|/|A|$. Show that the exponent 3 in the inequality $|A + A + A| \le K^3|A|$ is tight. (Hint: Consider the set $B = [m]^3 \cup \{(t,0,0) : t \in [n]\} \cup \{(0,t,0) : t \in [n]\} \cup \{(0,0,t) : t \in [n]\}$ inside $\mathbb{Z}^3$)

*This homework is from `http://www.borisbukh.org/DiscreteMath16/hw8.pdf`.