Walk through Combinatorics: homework #7*
Due 30 November 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. [2+(1 bonus)] Pick any two of the past homework problems from this semester. Generalize the statements and provide solutions. (Bonus will be awarded for particularly nice generalizations.)

2. [1+1+(1 bonus)] Let G be a graph, and let H be a graph obtained from G by deleting each edge with probability $1/2$. Let $\chi(G)$ and $\chi(H)$ be the chromatic numbers of G and H.

   (a) Show that $\mathbb{E}[\chi(H)] \geq \chi(G)^{1/2}$. (Hint: consider the complement of H).

   (b) Show that $\Pr[\chi(H) < c\chi(G)^{1/2}] \leq f(c)$ for an explicit function f satisfying $f(c) \to 0$ as $c \to 0$.

   (c) Let H' be a graph obtained from G by deleting each edge with probability $2/3$. Show that $\Pr[\chi(H') < c\chi(G)^{1/3}] \leq f'(c)$ for an explicit function f' satisfying $f(c) \to 0$ as $c \to 0$.

3. [2] A path of even length $P = v_1 v_2 \cdots v_{2k}$ in a graph (V, E) with a vertex coloring $f\colon V \to [r]$ is *periodic* if $f(v_j) = f(v_{j+k})$ for all j satisfying $1 \leq j \leq k$. Prove that there exists a constant r such that every graph G with maximum degree 5 admits a vertex r-coloring in which no path of of any even length is periodic.

*This homework is from `http://www.borisbukh.org/DiscreteMath16/hw7.pdf`.

4. [2] For a continuous function $f\colon \mathbb{R} \to [0,1]$ and a bounded interval $I \subset \mathbb{R}$, we define $N(I) = \int_I f(x)\,dx$ and $d(I) = N(I)/\operatorname{len}(I)$, where $\operatorname{len}(I)$ is the length of I. The interval I is called *ε-regular* if for every subinterval I' of length at least $\varepsilon \cdot \operatorname{len}(I')$ we have $|d(I) - d(I')| \le \varepsilon$.

   Show that for every $\varepsilon > 0$ there exists a number $M(\varepsilon)$ with the following property: For every continuous function $f\colon \mathbb{R} \to [0,1]$ there exists a partition of the interval $[0,1]$ into intervals $I_1, \dots, I_r$ with $r \le M(\varepsilon)$ such that

$$\sum_{I_t \text{ is } \varepsilon\text{-regular}} \operatorname{len}(I_t) \ge 1 - \varepsilon.$$

5. [2] Show that there exists a function $t(n)$ satisfying $t(n) = o(\sqrt{n})$ such that for each n there exists an interval I_n of length $t(n)$ satisfying the following: If G is a graph chosen uniformly at random among all graphs on the vertex set $[n]$, then $\Pr\bigl[\chi(G) \in I_n\bigr] \ge 0.99$. (Note that we proved a weaker result in class.)

6. (Bonus problem) Enjoy Thanksgiving!