Walk through Combinatorics: homework #6*
Due 16 November 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. [2] Show that that there exists a constant c with the following property. Whenever G is a graph on n vertices and $\frac{1}{2}\binom{n}{2}$ edges, then there is a subset U of vertices such that $|e(U) - \frac{1}{2}\binom{|U|}{2}| \geq cn^{3/2}$.

2. [1] Let A be a random subset of $[n]$ such that $\Pr[t \in A] = p$ and the events $t \in [n]$ are independent. Let E denote the event that A contains a 4-term arithmetic progression.

   Find a function $t(n)$ such that

   - if $p = o(t(n))$, then $\Pr[E] \to 0$; and
   - if $p = w(t(n))$, then $\Pr[E] \to 1$.

3. [2] Let $0 < \alpha < 1$ be a rational number, and let m be a natural number, and let $\varepsilon > 0$ be arbitrary. Show that for all $n > n_0(\alpha, \varepsilon, m)$ satisfying $\alpha n \in \mathbb{Z}$ there exist sets $A_1, \ldots, A_m \subset [n]$ of size αn each such that $|A_i \cap A_j| \leq (\alpha^2 + \varepsilon)n$ for all distinct $i, j \in [m]$.

4. [2] Write down an explicit positive constant c such that the following holds for all natural numbers n. Suppose $a_1, \ldots, a_n$ are n real numbers satisfying $\sum_{i=1}^n a_i^2 = 1$, and $\epsilon_1, \ldots, \epsilon_n \in \{-1, +1\}$ are uniformly and independently chosen signs. Then
$$\Pr\left[\left|\sum_{i=1}^n \epsilon_i a_i\right| \leq 1\right] \geq c.$$

*This homework is from `http://www.borisbukh.org/DiscreteMath16/hw6.pdf`.

5. [2 extra credit + much extra credit]

   (a) Let $\chi(G)$ denote the chromatic number of a graph G. Show that for each r there exists a number $M = M(r)$ such that every graph G of chromatic number M contains a subgraph G' with $\chi(G') \geq r$ and the girth at least 4.

   (b) (Open problem) Show the same with "girth at least 4" replaced by "girth at least 5".