Walk through Combinatorics: homework #4*
Due 19 October 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. [1] Show that $n! \geq (n/e)^n$ by estimating $[z^n]e^z$ via Cauchy's inequality for the complex integral.

2. [1+1] Prove the identity
$$\sum_k k\binom{n}{k} = n2^{n-1}$$
in two ways: by considering an appropriate power series, and by exhibiting a bijection between objects counted by the two sides of the equation.

3. [2+1] Let F_n be the number of functions $[n] \to \mathbb{N}$ whose image is $[r]$ for some r.
   (a) Find the exponential generating function for F_n. (Hint: consider individual values of r.)
   (b) Find an asymptotics for F_n. (Your answer should be of the form $F_n = A_n\bigl(1 + O(c^n)\bigr)$ for some explicit A_n and an explicit value of $c < 1$.)

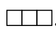
4. [2+(2 extra credit)]
   (a) Let $m \in \mathbb{N}$ be arbitrary fixed number. Let S_n be the number of sequences of $+1$ and -1 of length n such that all the partial sums lie in the intervals $[-m, m]$ and the total sum is zero. Find a closed form for the generating function $\sum S_n z^n$. [A "closed form" here means an expression in z and m

*This homework is from `http://www.borisbukh.org/DiscreteMath16/hw4.pdf`.

that involves only operations of addition, multiplication, exponentiation and their inverses.]

(b) Show that $S_n = (c_m + o(1))^n$ and determine an asymptotics for $|2 - c_m|$ as $m \to \infty$.

5. [2] Let $A \subset \mathbb{N}$ be an infinite set of integers. Let R_n be the number of ways to write n as $n = a_1 + a_2$ with $a_1, a_2 \in A$, where the order of summands does not matter (i.e. $5 = 3 + 2$ and $5 = 2 + 3$ are the same). Show that $R_n = C$ cannot hold for all $n \geq n_0$, for any constants C and n_0. (Hint: singularities of the generating function.)

6. [2] A *trimino* consists of three connected unit squares. In other words, the triminos are ,  and whatever can be obtained from these by rotation and reflection. Let T_n be the number of ways to cut 3-by-n rectangle into triminos. Find a closed form for $f(z) = \sum T_n z^n$.

LaTeX tip: The in-line pictures above were produced by TikZ package and code

```
\tikz[scale=0.2]{
      \draw (0,0) rectangle (1,1);
      \draw (1,0) rectangle (2,1);
      \draw (2,0) rectangle (3,1);}
```

and

```
\tikz[scale=0.2]{
      \draw (0,0) rectangle (1,1);
      \draw (1,0) rectangle (2,1);
      \draw (1,1) rectangle (2,2);}
```

respectively.