Walk through Combinatorics: homework #3*
Due 5 October 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. (a) [2] Show that for each n and r there exists N with the following property: For every coloring $\chi\colon \binom{[N]}{2} \times \binom{[N]}{2} \to [r]$ of ordered pairs of edges of K_N, there are sets $X, Y \subset [N]$ of size $|X| = |Y| = n$ such that χ is monochromatic on $\binom{X}{2} \times \binom{Y}{2}$.

   (b) [1] Prove or disprove that for every coloring $\chi\colon \binom{\mathbb{N}}{2} \times \binom{\mathbb{N}}{2} \to [r]$ there exist infinite sets $X, Y \subset \mathbb{N}$ such that χ is monochromatic on $\binom{X}{2} \times \binom{Y}{2}$.

2. [1] Prove that for each coloring $\chi\colon \binom{\mathbb{N}}{2} \to \mathbb{N}$ there exists an infinite subset $X \subset \mathbb{N}$ such that either

   (a) χ is monochromatic on $\binom{X}{2}$, or

   (b) no two edges in $\binom{X}{2}$ receive the same color, or

   (c) $\chi(\{i, j\})$ depends only on $\min(i, j)$, or

   (d) $\chi(\{i, j\})$ depends only on $\max(i, j)$.

3. (a) [1] Show that for each $\alpha > 0$ there exists $\beta > 0$ and n_0 such that whenever $n \geq n_0$ and $A \subset [n]$ is of density at least α, the number of 3-APs in A is at least βn^2.

   (b) [1] A *d-Roth-cube* is a set of the form $x_0 + x_1 \cdot \{0, 1, 2\} + \cdots + x_d \cdot \{0, 1, 2\}$ where $x_1, \ldots, x_d$ are all non-zero. Deduce from part (a) that for every $\alpha > 0$ and $d \in \mathbb{N}$ there exists $n = n(\alpha, d)$ such that every set $A \subset [n]$ of

*This homework is from `http://www.borisbukh.org/DiscreteMath16/hw3.pdf`.

density at least α contains a d-Roth-cube. (You may do this even if you did not solve part (a).)

4. (a) [2] Show that for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ with the following property. Whenever α is a real number there are integers q and p such that $1 \le q \le N$ and
$$|q^2\alpha - p| \le \varepsilon.$$
(Hint: van der Waerden's theorem.)

   (b) (Open problem; much extra credit) May one take $N = \varepsilon^{-1+o(1)}$ in the above?

5. [2+1] For a coloring $\chi\colon [n] \to [r]$ a set $X \subset [n]$ is *rainbow* if all the elements of X receive different colors.

   (a) Show that for each k there is a $c = c(k) > 0$ such that if $[n]$ is colored in any number of colors, and no color occurs more than cn times, then there is a rainbow arithmetic progression of length k.

   (b) Deduce, using Szemerédi's theorem, that for each k there an n such that if $[n]$ is colored (in any number of colors), then there is a non-trivial arithmetic progression of length k that is either monochromatic or rainbow.

6. [2+(1 extra credit)] The *step* of an arithmetic progression $\{a, a + d, a + 2d, \dots, a + (k-1)d\}$ is defined to be $|d|$, the Euclidean norm of d.

   (a) Show that there is an r such that for each n there is a coloring $\chi\colon \mathbb{R}^n \to [r]$ that contains no monochromatic 3-AP with step 1. [Hint: choose $\chi(x)$ that depends only on $|x|$.]

   (b) [Extra credit] Show that for every r there is an n such that for every r-coloring χ of $\mathbb{R}^n$ there is a 4-AP $\{x_1, x_2, x_3, x_4\}$ with step 1 that satisfies $\chi(x_1) = \chi(x_4)$ and $\chi(x_2) = \chi(x_3)$. Here it is understood that x_1, x_2, x_3, x_4 are in order, i.e., they satisfy $2x_2 = x_1 + x_3$ and $2x_3 = x_2 + x_4$. [Hint: Consider the coloring of $\{0, \lambda, 2\lambda, 3\lambda\}^n \subset \mathbb{R}^n$, for a suitable λ.]