Walk through Combinatorics: homework #1*
Due 7 September 2016, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [1] Give an alternative proof of the Erdős–Szekeres theorem using induction.

2. [1+1]

   (a) Let r and b be some fixed natural numbers. Edges of a complete graph on $[n] = \{1, 2, \ldots, n\}$ are colored red and blue. Show that if n is large enough, then the graph contains either an r-vertex red clique or a blue path of length b whose vertices are in increasing order.

   (b) Find the smallest n such that (a) holds.

3. [1+1] Let T be a tree having k vertices (a *tree* is a connected graph containing no cycles)

   (a) Prove that if $n > (k-1)(l-1)$ and K_n is colored red/blue, then K_n contains a red T or a blue K_l.

   (b) Show that this is not true if $n = (k-1)(l-1)$.

4. For a function $\phi\colon \mathbb{R} \to \mathbb{R}$ and a k-tuple $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ we define $\phi(x) = \big(\phi(x_1), \ldots, \phi(x_k)\big)$.

   Consider sequences with entries in $\mathbb{R}^k$. Call two such sequences $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ *isomorphic* if there is an order-preserving bijection $\phi\colon \mathbb{R} \to \mathbb{R}$ such that $\phi(a_i) = b_i$ for all i. Call a sequence $a_1, \ldots, a_n$ *homogeneous* if there is an arbitrarily long sequences all of whose n-element subsequences are isomorphic to $a_1, \ldots, a_n$

*This homework is from `http://www.borisbukh.org/DiscreteMath16/hw1.pdf`.

(a) [0, do not have to turn in] Find all homogeneous sequences for the case $k = 1$.

(b) [1 extra credit] Show that for each k, there is $f(k)$ (which depends only on k) such that the number of isomorphism classes of homogeneous sequences of length n is at most $f(k)$.