Walk through Combinatorics: homework #5*
Due 9 November 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in LaTeX via e-mail. I want both the LaTeX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. [Two directions: 1+1] *Sign* of a real number x is defined by

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

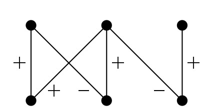
The definition extends to real matrices in the natural way: If M is a real matrix, then $\operatorname{sgn} M$ is a matrix given by $(\operatorname{sgn} M)_{i,j} = \operatorname{sgn} M_{i,j}$.

Let G be a bipartite graph that contains a perfect matching. Let B be its bipartite adjacency matrix, and σ a signing of G. Prove that the following two conditions are equivalent:

(a) σ is a Kasteleyn signing of G;

(b) Every matrix M satisfying $\operatorname{sgn} M = B^\sigma$ is a non-singular matrix.

For example, the above asserts that the "reason" why every matrix of the form $\left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & e & f \end{smallmatrix}\right)$ where $a, b, d, f > 0$ and $c, e < 0$ is non-singular is that



is a Kasteleyn signing.

*This homework is from `http://www.borisbukh.org/DiscreteMath15/hw5.pdf`.

2. [2] Denote by $M(G)$ the number of perfect matchings in a graph G. Show that if G is a connected graph with n vertices and $n-1+k$ edges, then $M(G) \leq f(k)$ for some explicit function f that is independent of n. [Hint: start with $k=0$.]

3. [2+1] Let $\mathcal{F} = \{(A_1, B_1), \ldots, (A_k, B_k)\}$ be a family of pairs of subsets of $[n]$ such that
$$A_i \cap B_j = \emptyset \iff i = j.$$

   (a) Prove that
$$\sum_{i=1}^{k} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \leq 1.$$

   (b) Give infinitely many examples of families $\mathcal{F}$ for which the inequality is sharp (i.e. the sum on the left-hand side is equal to 1).

4. [2] Show that that there exists a constant c with the following property. Whenever G is a graph on n vertices and $\frac{1}{2}\binom{n}{2}$ edges, then there is a subset U of vertices such that $|e(U) - \frac{1}{2}\binom{|U|}{2}| \geq cn^{3/2}$.

5. [0+(1 extra credit)]

   (a) Problem removed from the homework.

   (b) Find a set A satisfying $cn^\alpha \leq |A \cap [n]|$ for all n, but for which no set B such that the number of solutions to
$$m = a + b, \quad a \in A,\ b \in B$$
is between 1 and $C \log m$ for all m.