

# Walk through Combinatorics: homework #4\*

## Due 21 October 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in L<sup>A</sup>T<sub>E</sub>X via e-mail. I want both the L<sup>A</sup>T<sub>E</sub>X file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

1. [1+1]

- (a) Show that  $n! \geq (n/e)^n$  by estimating  $[z^n]e^z$  via Cauchy's inequality for the complex integral.
- (b) Problem removed from the homework.

2. [1+1] Prove the identity

$$\sum_k k \binom{n}{k} = n2^{n-1}$$

in two ways: by considering an appropriate power series, and by exhibiting a bijection between objects counted by the two sides of the equation.

3. [2+(2 extra credit)]

- (a) Let  $m \in \mathbb{N}$  be arbitrary fixed number. Let  $S_n$  be the number of sequences of  $+1$  and  $-1$  of length  $n$  such that all the partial sums lie in the intervals  $[-m, m]$  and the total sum is zero. Find a closed form for the generating function  $\sum S_n z^n$ . [A “closed form” here means an expression in  $z$  that involves only operations of addition, multiplication, exponentiation and their inverses.]
- (b) Show that  $S_n = (c_m + o(1))^n$  and determine an asymptotics for  $|2 - c_m|$  as  $m \rightarrow \infty$ .

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\*This homework is from <http://www.borisbukh.org/DiscreteMath15/hw4.pdf>.

4. [2] A *trimino* consists of three connected unit squares. In other words, the triminos are  $\square\square\square$ ,  $\square\begin{smallmatrix} \square \\ \square \end{smallmatrix}$  and whatever can be obtained from these by rotation and reflection. Let  $T_n$  be the number of ways to cut 3-by- $n$  rectangle into triminos. Find a closed form for  $f(z) = \sum T_n z^n$ .

L<sup>A</sup>T<sub>E</sub>X tip: The in-line pictures above were produced by TikZ package and code

```
\tikz[scale=0.2]{
  \draw (0,0) rectangle (1,1);
  \draw (1,0) rectangle (2,1);
  \draw (2,0) rectangle (3,1);}
```

and

```
\tikz[scale=0.2]{
  \draw (0,0) rectangle (1,1);
  \draw (1,0) rectangle (2,1);
  \draw (1,1) rectangle (2,2);}
```

respectively.