

Walk through Combinatorics: homework #2*

Due 23 September 2015, at start of class

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. For your own learning, you are advised to work individually. Collaboration may involve only discussion; all the writing must be done individually.

Homework must be submitted in L^AT_EX via e-mail. I want both the L^AT_EX file and the resulting PDF. The files must be of the form `lastname_discr_hwnum.tex` and `lastname_discr_hwnum.pdf` respectively. Pictures do not have to be typeset; a legible photograph of a hand-drawn picture is acceptable.

- [1] Show that if \mathbb{N} is colored into finitely many colors, then there are three *distinct* natural numbers x, y, z of the same color such that $x + y = z$.
- [2] Let $k > r$ be natural numbers. Consider an underdetermined system of homogeneous linear equations

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k &= 0, \\a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k &= 0, \\&\vdots \\a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rk}x_k &= 0\end{aligned}$$

with coefficients a_{ij} that are integers and satisfy $|a_{ij}| \leq N$ for all i and j . Show that there is a solution such that all the unknowns x_1, \dots, x_k are integers that are not all zero and that satisfy $|x_j| \leq C_{k,r}N^{r/(k-r)}$. Here $C_{k,r}$ is a constant that depends only on k and r . You do not have to find the smallest $C_{k,r}$ for which this holds.

- [2] Product of graphs G_1, \dots, G_n is the graph whose vertices are ordered tuples (v_1, \dots, v_n) s.t. $v_i \in G_i$, and the pair $(a_1, \dots, a_n), (b_1, \dots, b_n)$ forms an edge if and only if for each i either $a_i = b_i$ or $a_i b_i \in E(G_i)$.

*This homework is from <http://www.borisbukh.org/DiscreteMath15/hw2.pdf>.

Let $\alpha(G)$ denote the size of the largest independent set in G . Prove that the Ramsey number $R(\underbrace{3, \dots, 3}_n)$ is equal to $1 + \max \alpha(G_1 \times \dots \times G_n)$ where the maximum is over all graphs G_1, \dots, G_n such that $\alpha(G) = 2$.

4. [2] Suppose $A_1, A_2, \dots, A_m \subset [n]$ are sets of size αn . Show that there exist distinct i, j such that $|A_i \cap A_j| \geq (\alpha^2 - \varepsilon)n$, where $\varepsilon = \varepsilon(\alpha, m)$ depends only on α and m , and which tends to 0 as $m \rightarrow \infty$. (Fact: the number α^2 in this problem cannot be made larger. You might want to show that, but it is not required.)
5. [2+(extra credit) 2] Let X be a set. A n -letter word over alphabet X is simply an element of X^n , i.e., a sequence of n elements from X . Consider a word $w \in ([k] \cup \{*\})^n$. Let $C(w)$ be the set of all the words in $[k]^n$ that can be obtained by replacing stars by elements of $[k]$. For example, if $k = 2$ then

$$C(1 * 21*) = \{11211, 11212, 12211, 12212\}.$$

In general, if w has m stars, then $|C(w)| = 2^m$.

- (a) Use Ramsey's theorem (or anything else) to show that there is a function $n_0(r, m)$ such that if $n \geq n_0(r, m)$ and $\chi: [2]^n \rightarrow [r]$ is a coloring of n -letter words over 2-letter alphabet, then there is a word $w \in ([2] \cup \{*\})^n$ with m stars such that for $w' \in C(w)$ the color $\chi(w')$ depends only on the number of occurrences of 1's and 2's in w' .
- (b) Show that the analogous statement for 3-letter alphabet is false. Namely, show that there is an r and m such that for every n there is a coloring $\chi: [3]^n \rightarrow [r]$ so that for no $w \in ([3] \cup \{*\})^n$ with m stars the color of a word in $C(w)$ depends only on the number of occurrences of 1's, 2's and 3's.