Walk through Combinatorics: homework #5*
Due 19 November 2014, at start of class

Collaboration and use of external sources are **forbidden**. If unsure, ask me first.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let r be a positive integer. Show that for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, r)$ such that if a graph G on n vertices cannot be made K_r-free by deleting εn^2 edges, then it contains at least δn^r copies of K_r.

2. [2] The *midpoint* of a line segment $[p, q]$ is the point $(p + q)/2$. Let $\varepsilon > 0$ be a fixed real number. Show that, for each n, there exists a set $A \subset \mathbb{R}^2$ of n points such that

   - No three points of A are collinear; and
   - The set of the midpoints of all the $\binom{n}{2}$ line segments spanned by A has only $O(n^{1+\varepsilon})$ elements. (The constant in the big-oh notation is allowed to depend on ε.)

3. [2] For a continuous function $f : \mathbb{R} \to [0, 1]$ and a bounded interval $I \subset \mathbb{R}$, we define $N(I) = \int_I f(x)\,dx$ and $d(I) = N(I)/\operatorname{len}(I)$, where $\operatorname{len}(I)$ is the length of I. The interval I is called *ε-regular* if for every subinterval I' of length at least $\varepsilon \cdot \operatorname{len}(I')$ we have $|d(I) - d(I')| \le \varepsilon$.

   Show that for every $\varepsilon > 0$ there exists a number $M(\varepsilon)$ with the following property: For every continuous function $f : \mathbb{R} \to [0, 1]$ there exists a partition of the interval $[0, 1]$ into intervals $I_1, \ldots, I_r$ with $r \le M(\varepsilon)$ such that

$$\sum_{I_t \text{ is } \varepsilon\text{-regular}} \operatorname{len}(I_t) \ge 1 - \varepsilon.$$

*This homework is from `http://www.borisbukh.org/DiscreteMath14/hw5.pdf`.

4. [1+1] Prove the identity

$$\sum_k k\binom{n}{k} = n2^{n-1}$$

in two ways: by considering an appropriate power series, and by exhibiting a bijection between objects counted by the two sides of the equation.

5. [2] Suppose the denominations of coins in Fictionland are $d_1, d_2, \ldots, d_k$, and gcd of any $k-1$ of the denominations is 1. Let A_n be the number of ways to change n cents. Find an asymptotic formula for A_n of the form

$$A_n = c_0 n^{k-1} + c_1 n^{k-2} + O(n^{k-3}),$$

where c_0 and c_1 are explicit constants given in terms of $d_1, \ldots, d_k$.