Walk through Combinatorics: homework #4*
Due 5 November 2014, at start of class

Collaboration and use of external sources is forbidden. If unsure, ask me first.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Show that for each \( r \) there is \( m = m(r) \) with the following property: If \( \mathcal{H} = (V, E) \) is an \( r \)-uniform hypergraph such that every vertex is in at least \( m \) edges, then it is possible to find a partition \( E = E_1 \cup E_2 \) such that for each vertex \( v \) there are \( e_1 \in E_1 \) and \( e_2 \in E_2 \) such that \( v \) is both in \( e_1 \) and \( e_2 \).

2. [1+1] Let \( G \) be a graph, and let \( H \) be a graph obtained from \( G \) by deleting each edge with probability \( 1/2 \). Let \( \chi(G) \) and \( \chi(H) \) be the chromatic numbers of \( G \) and \( H \).

   (a) Show that \( \mathbb{E}[\chi(H)] \geq \chi(G)^{1/2} \). (Hint: consider the complement of \( H \)).

   (b) Show that \( \Pr[\chi(H) < c\chi(G)^{1/2}] \leq f(c) \) for an explicit function \( f \) satisfying \( f(c) \to 0 \) as \( c \to 0 \).

3. [2] A path of even length \( P = v_1v_2 \cdots v_{2k} \) in a graph \( (V, E) \) with a vertex coloring \( f: V \to [r] \) is periodic if \( f(v_j) = f(v_{j+k}) \) for all \( j \) satisfying \( 1 \leq j \leq k \). Prove that there exists a constant \( r \) such that every graph \( G \) with maximum degree 5 admits a vertex \( r \)-coloring in which no path of of any even length is periodic.

4. [2] For a permutation \( \pi \), let \( X = X(\pi) \) be the least number \( m \) such that \( \pi \) is a product of \( m \) cycles. Show that there is a constant \( C \) such that if one picks \( \pi \) uniformly at random from the symmetric group \( S_n \), then

\[
\Pr[|X - \mathbb{E}[X]| \geq C\sqrt{n}] \leq 0.01
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5. [2] Show that there exists a function $t(n)$ satisfying $t(n) = o(\sqrt{n})$ such that for each $n$ there exists an interval $I_n$ of length $t(n)$ satisfying the following: If $G$ is a graph chosen uniformly at random among all graphs on the vertex set $[n]$, then $\Pr[\chi(G) \in I_n] \geq 0.99$. (Note that we proved a weaker result in class.)