Walk through Combinatorics: homework #4*
Due 5 November 2014, at start of class

Collaboration and use of external sources is **forbidden**. If unsure, ask me first.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Show that for each r there is $m = m(r)$ with the following property: If $\mathcal{H} = (V, E)$ is an r-uniform hypergraph such that every vertex is in at least m edges, then it is possible to find a partition $E = E_1 \cup E_2$ such that for each vertex v there are $e_1 \in E_1$ and $e_2 \in E_2$ such that v is both in e_1 and e_2.

2. [1+1] Let G be a graph, and let H be a graph obtained from G by deleting each edge with probability $1/2$. Let $\chi(G)$ and $\chi(H)$ be the chromatic numbers of G and H.

   (a) Show that $\mathbb{E}[\chi(H)] \geq \chi(G)^{1/2}$. (Hint: consider the complement of H).

   (b) Show that $\Pr[\chi(H) < c\chi(G)^{1/2}] \leq f(c)$ for an explicit function f satisfying $f(c) \to 0$ as $c \to 0$.

3. [2] A path of even length $P = v_1 v_2 \cdots v_{2k}$ in a graph (V, E) with a vertex coloring $f\colon V \to [r]$ is *periodic* if $f(v_j) = f(v_{j+k})$ for all j satisfying $1 \leq j \leq k$. Prove that there exists a constant r such that every graph G with maximum degree 5 admits a vertex r-coloring in which no path of of any even length is periodic.

4. [2] For a permutation π, let $X = X(\pi)$ be the least number m such that π is a product of m cycles. Show that there is a constant C such that if one picks π uniformly at random from the symmetric group S_n, then

$$\Pr\left[|X - \mathbb{E}[X]| \geq C\sqrt{n}\right] \leq 0.01$$

*This homework is from `http://www.borisbukh.org/DiscreteMath14/hw4.pdf`.

5. [2] Show that there exists a function $t(n)$ satisfying $t(n) = o(\sqrt{n})$ such that for each n there exists an interval I_n of length $t(n)$ satisfying the following: If G is a graph chosen uniformly at random among all graphs on the vertex set $[n]$, then $\Pr\big[\chi(G) \in I_n\big] \geq 0.99$. (Note that we proved a weaker result in class.)