# Walk through Combinatorics: homework #3*
# Due 22 October 2014, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [1] Let $A$ be a random subset of $[n]$ such that $\Pr[t \in A] = p$ and the events $t \in [n]$ are independent. Let $E$ denote the event that $A$ contains a 4-term arithmetic progression.

   Find a function $t(n)$ such that

   - if $p = o(t(n))$, then $\Pr[E] \to 0$; and
   - if $p = w(t(n))$, then $\Pr[E] \to 1$.

2. [2] Let $0 < \alpha < 1$ be a rational number, let $\varepsilon > 0$ be arbitrary, and let $m$ be a fixed natural number. Show that for all $n > n_0(\alpha, \varepsilon, m)$ satisfying $\alpha n \in \mathbb{Z}$ there exist sets $A_1, \ldots, A_m \subset [n]$ of size $\alpha n$ each such that $|A_i \cap A_j| \leq (\alpha^2 + \varepsilon)n$ for all distinct $i, j \in [m]$.

3. [1] Suppose $X$ is a random variables taking nonnegative integer values. Show that
$$\Pr[X = 0] \leq \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}$$
   (You may assume that $\mathbb{E}[X^2]$ is non-zero, and is finite.)

4. [2] Write down an explicit positive constant $c$ such that the following holds for all natural numbers $n$. Suppose $a_1, \ldots, a_n$ are $n$ real numbers satisfying $\sum_{i=1}^n a_i^2 = 1$, and $\epsilon_1, \ldots, \epsilon_n \in \{-1, +1\}$ are uniformly and independently chosen signs. Then
$$\Pr\left[\left|\sum_{i=1}^n \epsilon_i a_i\right| \leq 1\right] \geq c.$$

*This homework is from `http://www.borisbukh.org/DiscreteMath14/hw3.pdf`.

5. [2] Let $\vec{v}_1 = (x_1, y_1), \ldots, \vec{v}_n = (x_n, y_n)$ be $n$ vectors in $\mathbb{Z}^2$ satisfying $|x_i|, |y_i| \leq \frac{2^{n/2}}{100\sqrt{n}}$. Show that there are two disjoint distinct sets $I, J \subset [n]$ such that

$$\sum_{i \in I} \vec{v}_i = \sum_{j \in J} \vec{v}_j.$$

6. [2 extra credit] Let $\chi(G)$ denote the chromatic number of a graph $G$. Show that for each $r$ there exists a number $M = M(r)$ such that every graph $G$ of chromatic number $M$ contains a subgraph $G'$ with $\chi(G') \geq r$ and the girth at least 4.