Walk through Combinatorics: homework #2*
Due 6 October 2014, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. (a) [2] Show that for each n and r there exists N with the following property: For every coloring $\chi\colon \binom{[N]}{2} \times \binom{[N]}{2} \to [r]$ of ordered pairs of edges of K_N, there are sets $X, Y \subset [N]$ of size $|X| = |Y| = n$ such that χ is monochromatic on $\binom{X}{2} \times \binom{Y}{2}$.

   (b) [1] Prove or disprove that for every coloring $\chi\colon \binom{\mathbb{N}}{2} \times \binom{\mathbb{N}}{2} \to [r]$ there exist infinite sets $X, Y \subset \mathbb{N}$ such that χ is monochromatic on $\binom{X}{2} \times \binom{Y}{2}$.

2. [2] Show that for each $\varepsilon > 0$ there exists $N = N(\varepsilon)$ with the following property. Whenever α is a real number there are integers q and p such that $1 \le q \le N$ and
$$|q^2\alpha - p| \le \varepsilon.$$
(Hint: van der Waerden's theorem.)

3. (a) [1] Show that for each $\alpha > 0$ there exists $\beta > 0$ and n_0 such that whenever $n \ge n_0$ and $A \subset [n]$ is of density at least α, the number of 3-APs in A is at least βn^2.

   (b) [1] A *d-Roth-cube* is a set of the form $x_0 + x_1 \cdot \{0,1,2\} + \cdots + x_d \cdot \{0,1,2\}$ where $x_1, \ldots, x_d$ are all non-zero. Deduce from part (a) that for every $\alpha > 0$ and $d \in \mathbb{N}$ there exists $n = n(\alpha, d)$ such that every set $A \subset [n]$ of density at least α contains a d-Roth-cube.

4. [2] Let $s, t \in \mathbb{N}$. The vertex set of a graph G is a disjoint union of infinitely many blocks, each block being a set of size t. Inside any set of s distinct

*This homework is from `http://www.borisbukh.org/DiscreteMath14/hw2.pdf`.

blocks there is an edge that goes between two different blocks. Show that in G there is an infinite path visiting no block more than once.

5. [2+(1 extra credit)] The *step* of an arithmetic progression $\{a, a+d, a+2d, \dots, a+(k-1)d\}$ is defined to be $|d|$, the Euclidean norm of d.

   (a) Show that there is an r such that for each n there is a coloring $\chi \colon \mathbb{R}^n \to [r]$ that contains no monochromatic 3-AP with step 1. [Hint: choose $\chi(x)$ that depends only on $|x|$.]

   (b) [Extra credit] Show that for every r there is an n such that for every r-coloring χ of $\mathbb{R}^n$ there is a 4-AP $\{x_1, x_2, x_3, x_4\}$ with step 1 that satisfies $\chi(x_1) = \chi(x_4)$ and $\chi(x_2) = \chi(x_3)$. Here it is understood that x_1, x_2, x_3, x_4 are in order, i.e., they satisfy $2x_2 = x_1 + x_3$ and $2x_3 = x_2 + x_4$. [Hint: Consider the coloring of $\{0, \lambda, 2\lambda, 3\lambda\}^n \subset \mathbb{R}^n$, for a suitable λ.]