Walk through Combinatorics: homework #1*
Due 22 September 2014, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [1] Show that if $\mathbb{N}$ is colored into finitely many colors, then there are three *distinct* natural numbers x, y, z of the same color such that $x + y = z$.

2. (a) [1] Show that for every k there is an n such that whenever subsets of $[n]$ are colored into k colors there are non-empty disjoint sets A, B such that the color of A, the color of B, and the color of $A \cup B$ are all the same.

   (b) [1 extra credit] Show that for every k there is an n such that whenever subsets of $[n]$ are colored into k colors there are non-empty disjoint sets A, B, C such that the colors of six sets $A, B, C, A \cup B, A \cup C, B \cup C$ are all the same.

3. (a) [2] Show that every sequence of distinct real numbers either contains an increasing subsequence of length $s+1$ or can be partitioned into at most s decreasing subsequences.

   (b) [1] Deduce Erdős–Szekeres theorem on monotone subsequence from the statement in part (a). (You may do this part without doing part (a)).

4. For a set A of integers, put $\Sigma(A) \stackrel{\text{def}}{=} \sum_{a \in A} a$. Let $S \subset [n]$ be a set of m integers.

   (a) [2] Show that if $m \geq \log_2 n + \log_2 \log_2 n + 1$ and n is sufficiently large, then S contains two non-empty disjoint subsets A_1, A_2 such that $\Sigma(A_1) = \Sigma(A_2)$.

*This homework is from `http://www.borisbukh.org/DiscreteMath14/hw1.pdf`.

(b) (Open problem; extra credit) Prove or disprove that there is a constant C such that if $m \geq C \log_2 n$, then S contains three non-empty disjoint subsets A_1, A_2, A_3 such that $\Sigma(A_1) = \Sigma(A_2) = \Sigma(A_3)$.

5. [2] Show that the two-color hypergraph Ramsey numbers for r-uniform hypergraphs satisfy $R_r(k,k) \leq \mathrm{tw}_r\big(O(k)\big)$, where tw is the tower function that is defined by $\mathrm{tw}_1(x) = x$, $\mathrm{tw}_h(x) = 2^{\mathrm{tw}_{h-1}(x)}$.

6. [2] Product of graphs $G_1, \ldots, G_n$ is the graph whose vertices are ordered tuples $(v_1, \ldots, v_n)$ s.t. $v_i \in G_i$, and the pair $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ forms an edge if and only if for each i either $a_i = b_i$ or $a_i b_i \in E(G_i)$.

   Let $\alpha(G)$ denote the size of the largest independent set in G. Prove that the Ramsey number $R(\underbrace{3, \ldots, 3}_{n})$ is equal to $1 + \max \alpha(G_1 \times \cdots \times G_n)$ where the maximum is over all graphs $G_1, \ldots, G_n$ such that $\alpha(G_i) = 2$ for all i.

7. [2] Suppose $A_1, A_2, \ldots, A_m \subset [n]$ are sets of size αn. Show that there exist distinct i, j such that $|A_i \cap A_j| \geq \big(\alpha^2 - \varepsilon\big)n$, where $\varepsilon = \varepsilon(\alpha, m)$ depends only on α and m, and which tends to 0 as $m \to \infty$. (Fact: the number α^2 in this problem cannot be made larger. You might want to show that, but it is not required.)