Walk through Combinatorics: homework #1*
Due 22 September 2014, at start of class

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [1] Show that if \( \mathbb{N} \) is colored into finitely many colors, then there are three distinct natural numbers \( x, y, z \) of the same color such that \( x + y = z \).

2. (a) [1] Show that for every \( k \) there is an \( n \) such that whenever subsets of \( [n] \) are colored into \( k \) colors there are non-empty disjoint sets \( A, B \) such that the color of \( A \), the color of \( B \), and the color of \( A \cup B \) are all the same.

(b) [1 extra credit] Show that for every \( k \) there is an \( n \) such that whenever subsets of \( [n] \) are colored into \( k \) colors there are non-empty disjoint sets \( A, B, C \) such that the colors of six sets \( A, B, C, A \cup B, A \cup C, B \cup C \) are all the same.

3. (a) [2] Show that every sequence of distinct real numbers either contains an increasing subsequence of length \( s + 1 \) or can be partitioned into at most \( s \) decreasing subsequences.

(b) [1] Deduce Erdős–Szekeres theorem on monotone subsequence from the statement in part (a). (You may do this part without doing part (a)).

4. For a set \( A \) of integers, put \( \Sigma(A) \overset{\text{def}}{=} \sum_{a \in A} a \). Let \( S \subset [n] \) be a set of \( m \) integers.

(a) [2] Show that if \( m \geq \log_2 n + \log_2 \log_2 n + 1 \) and \( n \) is sufficiently large, then \( S \) contains two non-empty disjoint subsets \( A_1, A_2 \) such that \( \Sigma(A_1) = \Sigma(A_2) \).

*This homework is from [http://www.borisbukh.org/DiscreteMath14/hw1.pdf](http://www.borisbukh.org/DiscreteMath14/hw1.pdf)
(b) (Open problem; extra credit) Prove or disprove that there is a constant $C$ such that if $m \geq C \log_2 n$, then $S$ contains three non-empty disjoint subsets $A_1, A_2, A_3$ such that $\Sigma(A_1) = \Sigma(A_2) = \Sigma(A_3)$.

5. [2] Show that the two-color hypergraph Ramsey numbers for $r$-uniform hypergraphs satisfy $R_r(k, k) \leq \text{tw}_r(O(k))$, where $\text{tw}$ is the tower function that is defined by $\text{tw}_1(x) = x$, $\text{tw}_h(x) = 2^{\text{tw}_{h-1}(x)}$.

6. [2] Product of graphs $G_1, \ldots, G_n$ is the graph whose vertices are ordered tuples $(v_1, \ldots, v_n)$ s.t. $v_i \in G_i$, and the pair $(a_1, \ldots, a_n)$, $(b_1, \ldots, b_n)$ forms an edge if and only if for each $i$ either $a_i = b_i$ or $a_i b_i \in E(G_i)$.

Let $\alpha(G)$ denote the size of the largest independent set in $G$. Prove that the Ramsey number $R(3, \ldots, 3)$ is equal to $1 + \max \alpha(G_1 \times \cdots \times G_n)$ where the maximum is over all graphs $G_1, \ldots, G_n$ such that $\alpha(G_i) = 2$ for all $i$.

7. [2] Suppose $A_1, A_2, \ldots, A_m \subset [n]$ are sets of size $\alpha n$. Show that there exist distinct $i, j$ such that $|A_i \cap A_j| \geq (\alpha^2 - \varepsilon)n$, where $\varepsilon = \varepsilon(\alpha, m)$ depends only on $\alpha$ and $m$, and which tends to 0 as $m \to \infty$. (Fact: the number $\alpha^2$ in this problem cannot be made larger. You might want to show that, but it is not required.)