Walk through Combinatorics: homework #5*
Due 18 November 2013

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Recall that a d-cube is a set of the form $x_0 + \{0, x_1\} + \cdots + \{0, x_d\}$. A d-cube is *proper* if it contains 2^d elements. Show that for each integer $d \geq 2$ there is a set $A \subset [n]$ that contains no proper d-cube, and is of size
$$|A| \geq \tfrac{1}{2} n^{1-d/(2^d-1)}.$$

2. [1+1] In both problems below, A is a set of n positive *real* numbers.
   (a) Suppose $\vec{t} = (t_1, \ldots, t_m) \in \mathbb{Z}^m$ is an integer vector such that $t_1 + \cdots + t_m \neq 0$. Call a set B $\vec{t}$-free if there are no $b_1, \cdots, b_m \in B$ satisfying
   $$t_1 b_1 + \cdots + t_m b_m = 0.$$
   Show that there is a constant $c > 0$ (depending only on t's) so that A contains a $\vec{t}$-free set B of size at least cn.
   (b) Show that A contains a $(1, 1, -1)$-free subset of *strictly* more than $n/3$ elements.

3. [1+1] Let G be a directed graph on n vertices of minimum outdegree d.

*This homework is from `http://www.borisbukh.org/DiscreteMath13/hw5.pdf`.

(a) Show that if $d > \log_2 n$, then G contains a simple even cycle.

(b) Show that if $d > \log_2 n - \frac{1}{10}\log_2\log_2 n$, and n is large enough, then G contains a simple even cycle.

4. [1] Let A be a random subset of $[n]$ such that $\Pr[t \in A] = p$ and the events $t \in [n]$ are independent. Let E denote the event that A contains a 4-term arithmetic progression.

   Find a function $t(n)$ such that

   - if $p = o(t(n))$, then $\Pr[E] \to 0$; and
   - if $p = w(t(n))$, then $\Pr[E] \to 1$.

5. [2] Show that there is a positive constant c such that the following holds. Suppose $a_1, \ldots, a_n$ are n real numbers satisfying $\sum_{i=1}^n a_i^2 = 1$, and $\epsilon_1, \ldots, \epsilon_n \in \{-1, +1\}$ are uniformly and independently chosen signs. Show that

$$\Pr\left[\left|\sum_{i=1}^{n} \epsilon_i a_i\right| \leq 1\right] \geq c.$$

   In your solution the constant c must be explicit, but not necessarily the best possible.

6. [2] Suppose k is a positive integer. Let $G = (V, E)$ be a cycle of length kn and let $V = V_1 \cup \cdots \cup V_n$ be a partition of its kn vertices into n pairwise disjoint subsets, each of cardinality k.

   Show that if $k = 1000$, then there is an independent set in G containing precisely one vertex from each V_i.