Walk through Combinatorics: homework #4*
Due 1 November 2013

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let D_n be the number of ways to place nonoverlapping dominos on a 2-by-n rectangular board. Note that we do not require that every square of the board is covered by a domino. Find the generating function for D_n.

2. [Two directions: 1+1] *Sign* of a real number x is defined by

$$\operatorname{sgn} x = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

The definition extends to real matrices in the natural way: If M is a real matrix, then $\operatorname{sgn} M$ is a matrix given by $(\operatorname{sgn} M)_{i,j} = \operatorname{sgn} M_{i,j}$.

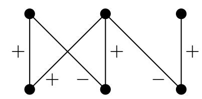
Let G be a bipartite graph that contains a perfect matching. Let B be its bipartite adjacency matrix, and σ a signing of G. Prove that the following two conditions are equivalent:

(a) σ is a Kasteleyn signing of G;

(b) Every matrix M satisfying $\operatorname{sgn} M = B^\sigma$ is a non-singular matrix.

*This homework is from `http://www.borisbukh.org/DiscreteMath13/hw4.pdf`.

For example, the above asserts that the "reason" why every matrix of the form $\left(\begin{smallmatrix} a & b & 0 \\ c & d & 0 \\ 0 & e & f \end{smallmatrix}\right)$ where $a, b, d, f > 0$ and $c, e < 0$ is non-singular is that



is a Kasteleyn signing.

3. [2] Denote by $M(G)$ the number of perfect matchings in a graph G. Show that if G is a connected graph with n vertices and $n-1+k$ edges, then $M(G) \le f(k)$ for some explicit function f that is independent of n. [Hint: start with $k = 0$.]

4. [2+1] Let $\mathcal{F} = \{(A_1, B_1), \ldots, (A_k, B_k)\}$ be a family of pairs of subsets of $[n]$ such that
$$A_i \cap B_j = \emptyset \iff i = j.$$

   (a) Prove that
$$\sum_{i=1}^{k} \frac{1}{\binom{|A_i|+|B_i|}{|A_i|}} \le 1.$$

   (b) Give infinitely many examples of families $\mathcal{F}$ for which the inequality is sharp (i.e. the sum on the left-hand side is equal to 1).