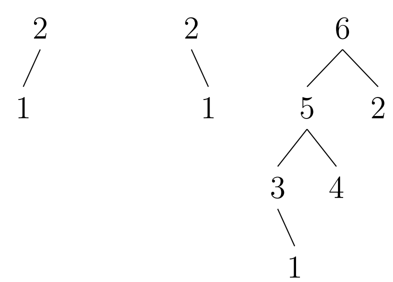
Walk through Combinatorics: homework #3[*]
Due 21 October 2013

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2+1] An *incomplete binary tree* is a rooted tree in which each node is either a terminal node, or a branch. A branch node might have either one child (left or right) or both left and right children. A *descending* labelling on a tree is an assignment of labels 1 through n to the nodes, where n is the number of nodes, such that on every path from the root the labels are descending. A *descending incomplete binary tree* is an incomplete binary tree with a descending labelling on it.



Three examples of descending incomplete binary trees

   (a) Use generating functions to compute the number of descending incomplete binary trees on n nodes.

[*]This homework is from `http://www.borisbukh.org/DiscreteMath13/hw3.pdf`.

(b) Give a bijective proof of the closed formula that you found in the part (a).

2. [2+1] Let F_n be the number of functions $[n] \to \mathbb{N}$ whose image is $[r]$ for some r.

   (a) Find the exponential generating function for F_n. (Hint: consider individual values of r.)

   (b) Find an asymptotics for F_n. (Your answer should be of the form $F_n = A_n\bigl(1 + O(c^n)\bigr)$ for some explicit A_n and an explicit value of $c < 1$.)

3. [2] Suppose $\{a_n\}$ is a sequence that satisfies a linear recurrence, i.e., a relation of the form $a_n = \sum_{k=1}^{d} \gamma_k a_{n-k}$ for some constants $\gamma_1, \dots, \gamma_d$. Let $b_n = a_{3n}$. Prove that b_n also satisfies a linear recurrence.

4. [2+(2 extra credit)] Let $\mathcal{S}_n$ be the set of all sequences of length $2n$ with n terms that are $+1$ and n terms that are -1, and all of whose partial sums are nonnegative. Let $C_{n,k}$ be the number sequences in $\mathcal{S}_n$ with exactly k zero partial sums. For example, the only sequence counted by $C_{2,2}$ is $+1, +1, -1, -1$.

   (a) Compute $F(z, w) = \sum C_{n,k} z^n w^k$ in closed form.

   (b) (Extra credit) Suppose we pick a sequence uniformly at random from $\mathcal{S}_n$. What is the expectation of the number of times a partial sum of coefficients vanishes? (You may want to use a symbolic algebra system for some computations.)