Walk through Combinatorics: homework #2*
Due 4 October 2013

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let $s, t \in \mathbb{N}$. The vertex set of a graph G is a disjoint union of infinitely many blocks, each block being a set of size t. Inside any set of s distinct blocks there is an edge that goes between two different blocks. Show that in G there is an infinite path visiting no block more than once.

2. [2] Let $\alpha, \beta > 0$ be arbitrary real numbers. Prove that there exists an N such that every $A \subset [N]$ of density at least α contains three numbers of the form $a, a + \lfloor \beta t \rfloor, a + \lfloor 2\beta t \rfloor$ for some integers a and $t > 0$

3. [2] Show the following strengthening of the multidimensional Hales–Jewett theorem: For every k, r, d there is an number N such that every coloring $\chi\colon [k]^N \to [r]$ there is a word $w \in \big([k] \cup \{*1, \dots, *d\}\big)^N$ such that
   - the d-dimensional combinatorial subspace associated to w is monochromatic in χ; and
   - if we denote by $P_i \subset [N]$ the set of positions at which $*i$ appears, then the sets $P_1, \dots, P_d$ are all translates of one another. [Hint: There is no need to reprove Hales–Jewett theorem.)

4. [2+1] For a coloring $\chi\colon [n] \to [r]$ a set $X \subset [n]$ is *rainbow* if all the elements of X receive different colors.

   (a) Show that for each k there is a $c = c(k) > 0$ such that if $[n]$ is colored in any number of colors, and no color occurs more than cn times, then there is a rainbow arithmetic progression of length k.

*This homework is from `http://www.borisbukh.org/DiscreteMath13/hw2.pdf`.

(b) Deduce, using Szemerédi's theorem, that for each k there an n such that if $[n]$ is colored (in any number of colors), then there is a non-trivial arithmetic progression of length k that is either monochromatic or rainbow.

5. [2+(1 extra credit)] The *step* of an arithmetic progression $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$ is defined to be $|d|$, the Euclidean norm of d.

   (a) Show that there is an r such that for each n there is a coloring $\chi\colon \mathbb{R}^n \to [r]$ that contains no monochromatic 3-AP with step 1. [Hint: choose $\chi(x)$ that depends only on $|x|$.]

   (b) [Extra credit] Show that for every r there is an n such that for every r-coloring χ of $\mathbb{R}^n$ there is a 4-AP $\{x_1, x_2, x_3, x_4\}$ with step 1 that satisfies $\chi(x_1) = \chi(x_4)$ and $\chi(x_2) = \chi(x_3)$. Here it is understood that x_1, x_2, x_3, x_4 are in order, i.e., they satisfy $2x_2 = x_1 + x_3$ and $2x_3 = x_2 + x_4$. [Hint: Consider the coloring of $\{0, \lambda, 2\lambda, 3\lambda\}^n \subset \mathbb{R}^n$, for a suitable λ.]

6. [Extra credit, 1+] Is it possible to strengthen Hales–Jewett theorem so that the set of stars is an AP? In other words, is there for each r and k a large N such that every coloring of $[k]^N$ contains a combinatorial line associated to a word $w \in ([k] \cup \{*\})^N$ in which positions of the stars form an AP?