Walk through Combinatorics: homework #2*
Due 4 October 2013

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let $s, t \in \mathbb{N}$. The vertex set of a graph $G$ is a disjoint union of infinitely many blocks, each block being a set of size $t$. Inside any set of $s$ distinct blocks there is an edge that goes between two different blocks. Show that in $G$ there is an infinite path visiting no block more than once.

2. [2] Let $\alpha, \beta > 0$ be arbitrary real numbers. Prove that there exists an $N$ such that every $A \subset [N]$ of density at least $\alpha$ contains three numbers of the form $a, a + \lfloor \beta t \rfloor, a + \lfloor 2\beta t \rfloor$ for some integers $a$ and $t > 0$.

3. [2] Show the following strengthening of the multidimensional Hales–Jewett theorem: For every $k, r, d$ there is an number $N$ such that every coloring $\chi : [k]^N \to [r]$ there is a word $w \in ([k] \cup \{*, 1, \ldots, d\})^N$ such that

- the $d$-dimensional combinatorial subspace associated to $w$ is monochromatic in $\chi$; and
- if we denote by $P_i \subset [N]$ the set of positions at which $*i$ appears, then the sets $P_1, \ldots, P_d$ are all translates of one another. [Hint: There is no need to reprove Hales–Jewett theorem.)

4. [2+1] For a coloring $\chi : [n] \to [r]$ a set $X \subset [n]$ is rainbow if all the elements of $X$ receive different colors.

(a) Show that for each $k$ there is a $c = c(k) > 0$ such that if $[n]$ is colored in any number of colors, and no color occurs more than $cn$ times, then there is a rainbow arithmetic progression of length $k$.

*This homework is from http://www.borisbukh.org/DiscreteMath13/hw2.pdf
(b) Deduce, using Szemerédi’s theorem, that for each $k$ there is an $n$ such that if $[n]$ is colored (in any number of colors), then there is a non-trivial arithmetic progression of length $k$ that is either monochromatic or rainbow.

5. [2+(1 extra credit)] The step of an arithmetic progression $\{a, a+d, a+2d, \ldots, a+(k-1)d\}$ is defined to be $|d|$, the Euclidean norm of $d$.

(a) Show that there is an $r$ such that for each $n$ there is a coloring $\chi: \mathbb{R}^n \to [r]$ that contains no monochromatic 3-AP with step 1. [Hint: choose $\chi(x)$ that depends only on $|x|$]

(b) [Extra credit] Show that for every $r$ there is an $n$ such that for every $r$-coloring $\chi$ of $\mathbb{R}^n$ there is a 4-AP $\{x_1, x_2, x_3, x_4\}$ with step 1 that satisfies $\chi(x_1) = \chi(x_4)$ and $\chi(x_2) = \chi(x_3)$. Here it is understood that $x_1, x_2, x_3, x_4$ are in order, i.e., they satisfy $2x_2 = x_1 + x_3$ and $2x_3 = x_2 + x_4$. [Hint: Consider the coloring of $\{0, \lambda, 2\lambda, 3\lambda\}^n \subset \mathbb{R}^n$, for a suitable $\lambda$.]

6. [Extra credit, 1+] Is it possible to strengthen Hales–Jewett theorem so that the set of stars is an AP? In other words, is there for each $r$ and $k$ a large $N$ such that every coloring of $[k]^N$ contains a combinatorial line associated to a word $w \in ([k] \cup \{\ast\})^N$ in which positions of the stars form an AP?