Walk through Combinatorics: homework #1*
Due 16 September 2013

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let $k, l, m$ be arbitrary natural numbers. Let $ES(k, l, m)$ be the length of the longest sequence of real numbers that contains neither

- a strictly increasing subsequence of length $k$, nor
- a strictly decreasing subsequence of length $l$, nor
- a constant subsequence of length $m$.

Find $ES(k, l, m)$.

2. [2] Let $k > r$ be natural numbers. Consider an underdetermined system of homogeneous linear equations

$$
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k = 0, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k = 0, \\
\vdots \hspace{2cm} \vdots \\
a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rk}x_k = 0
$$

with coefficients $a_{ij}$ that are integers and satisfy $|a_{ij}| \leq N$ for all $i$ and $j$. Show that there is a solution such that all the unknowns $x_1, \ldots, x_k$ are integers that are not all zero and that satisfy $|x_j| \leq C_{k,r} N^{r/(k-r)}$. Here $C_{k,r}$ is a constant that depends only on $k$ and $r$. You do not have to find the smallest $C_{k,r}$ for which this holds.

3. [2] Product of graphs $G_1, \ldots, G_n$ is the graph whose vertices are ordered tuples $(v_1, \ldots, v_n)$ s.t. $v_i \in G_i$, and the pair $(a_1, \ldots, a_n), (b_1, \ldots, b_n)$ forms an edge if and only if for each $i$ either $a_i = b_i$ or $a_ib_i \in E(G_i)$.

Let $\alpha(G)$ denote the size of the largest independent set in $G$. Prove that the Ramsey number $R(3, \ldots, 3)_{\leq n}$ is equal to $1 + \max\limits_{\alpha(G_1 \times \cdots \times G_n)} \alpha(G_1 \times \cdots \times G_n)$ where the maximum is over all graphs $G_1, \ldots, G_n$ such that $\alpha(G) = 2$.

4. [2+1] Let $T$ be a tree having $k$ vertices (a tree is a connected graph containing no cycles)

(a) If $n > (k-1)(l-1)$ and $K_n$ is colored red/blue, then $K_n$ contains a red $T$ or a blue $K_l$.

(b) Show that this is not true if $n = (k-1)(l-1)$.

5. [2] Let $r \in \mathbb{N}$ be any fixed integer. Use probabilistic method to give a lower bound on $R(3, \ldots, 3)_{\leq k}$. Your answer should be an order-of-magnitude asymptotics for a fixed $r$ and large $k$ (i.e., is your bound exponential, doubly-exponential, or of an intermediate growth rate?).

6. [Extra credit, 2+2] Let $X$ be a set. A $n$-letter word over alphabet $X$ is simply an element of $X^n$, i.e., a sequence of $n$ elements from $X$. Consider a word $w \in ([k] \cup \{\ast\})^n$. Let $C(w)$ be the set of all the words in $[k]^n$ that can be obtained by replacing stars by elements of $[k]$. For example, if $k = 2$ then

$$C(1 \ast 21\ast) = \{11211, 11212, 12211, 12212\}.$$ 

In general, if $w$ has $m$ stars, then $|C(w)| = 2^m$.

(a) Use Ramsey’s theorem (or anything else) to show that there is a function $n_0(r, m)$ such that if $n \geq n_0(r, m)$ and $\chi : [2]^n \rightarrow [r]$ is a coloring of $n$-letter words over 2-letter alphabet, then there is a word $w \in (\{2\} \cup \{\ast\})^n$ with $m$ stars such that for $w' \in C(w)$ the color $\chi(w')$ depends only on the number of occurrences of 1’s and 2’s in $w'$.

(b) Show that the analogous statement for 3-letter alphabet is false. Namely, show that there is an $r$ and $m$ such that for every $n$ there is a coloring $\chi : [3]^n \rightarrow [r]$ so that for no $w \in ([3] \cup \{\ast\})^n$ with $m$ stars the color of a word in $C(w)$ depends only on the number of occurrences of 1’s, 2’s and 3’s.