Walk through Combinatorics: homework #1*
Due 16 September 2013

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let k, l, m be arbitrary natural numbers. Let $ES(k, l, m)$ be the length of the longest sequence of real numbers that contains *neither*

   - a strictly increasing subsequence of length k, *nor*
   - a strictly decreasing subsequence of length l, *nor*
   - a constant subsequence of length m.

   Find $ES(k, l, m)$.

2. [2] Let $k > r$ be natural numbers. Consider an underdetermined system of homogeneous linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k &= 0, \\ \vdots \qquad\qquad & \vdots \\ a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rk}x_k &= 0 \end{aligned}$$

   with coefficients a_{ij} that are integers and satisfy $|a_{ij}| \le N$ for all i and j. Show that there is a solution such that all the unknowns $x_1, \ldots, x_k$ are integers that are not all zero and that satisfy $|x_j| \le C_{k,r} N^{r/(k-r)}$. Here $C_{k,r}$ is a constant that depends only on k and r. You do not have to find the smallest $C_{k,r}$ for which this holds.

*This homework is from `http://www.borisbukh.org/DiscreteMath13/hw1.pdf`.

3. [2] Product of graphs $G_1, \dots, G_n$ is the graph whose vertices are ordered tuples $(v_1, \dots, v_n)$ s.t. $v_i \in G_i$, and the pair $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ forms an edge if and only if for each i either $a_i = b_i$ or $a_i b_i \in E(G_i)$.

   Let $\alpha(G)$ denote the size of the largest independent set in G. Prove that the Ramsey number $R(\underbrace{3, \dots, 3}_{n})$ is equal to $1 + \max \alpha(G_1 \times \dots \times G_n)$ where the maximum is over all graphs $G_1, \dots, G_n$ such that $\alpha(G) = 2$.

4. [2+1] Let T be a tree having k vertices (a *tree* is a connected graph containing no cycles)

   (a) If $n > (k-1)(l-1)$ and K_n is colored red/blue, then K_n contains a red T or a blue K_l.

   (b) Show that this is not true if $n = (k-1)(l-1)$.

5. [2] Let $r \in \mathbb{N}$ be any fixed integer. Use probabilistic method to give a lower bound on $R_3(\overbrace{k, \dots, k}^{r})$. Your answer should be an order-of-magnitude asymptotics for a fixed r and large k (i.e., is your bound exponential, doubly-exponential, or of an intermediate growth rate?).

6. [Extra credit, 2+2] Let X be a set. A *n-letter word over alphabet X* is simply an element of X^n, i.e., a sequence of n elements from X. Consider a word $w \in \big([k] \cup \{*\}\big)^n$. Let $C(w)$ be the set of all the words in $[k]^n$ that can be obtained by replacing stars by elements of $[k]$. For example, if $k = 2$ then

   $$C(1 * 21*) = \{11211, 11212, 12211, 12212\}.$$

   In general, if w has m stars, then $|C(w)| = 2^m$.

   (a) Use Ramsey's theorem (or anything else) to show that there is a function $n_0(r, m)$ such that if $n \geq n_0(r, m)$ and $\chi \colon [2]^n \to [r]$ is a coloring of n-letter words over 2-letter alphabet, then there is a word $w \in ([2] \cup \{*\})^n$ with m stars such that for $w' \in C(w)$ the color $\chi(w')$ depends only on the number of occurences of 1's and 2's in w'.

   (b) Show that the analogous statement for 3-letter alphabet is false. Namely, show that there is an r and m such that for every n there is a coloring $\chi \colon [3]^n \to [r]$ so that for no $w \in ([3] \cup \{*\})^n$ with m stars the color of a word in $C(w)$ depends only on the number of occurences of 1's, 2's and 3's.