Walk through Combinatorics: Compactness principle*

There are many statements in mathematics that go under the name of "compactness principle". Their common property is their ability to transfer the statements about finite objects to the statements about infinite objects. Also, many of them are in some way related to Tychonoff's compactness theorem. The following combinatorial compactness principle is one of them. In our exposition we will not use Tychonoff's theorem, but Zorn's lemma[1].

A *proper (vertex) r-coloring* of a hypergraph $H = (V, E)$ is a map $\chi\colon V \to [r]$ such that no edge $e \in E$ is monochromatic. Given a subset $W \subset V$ of the vertices an *induced hypergraph* on W, denoted by $H[W]$, is a hypergraph whose vertex set is W and whose edges are those edges of H that are contained in W.

Theorem 1 (Compactness principle). *Suppose all the edges of a hypergraph $H = (V, E)$ are finite. If for every finite set $W \subset V$ the induced hypergraph $H[W]$ is properly r-colorable, then H itself is properly r-colorable.*

Note that the compactness principle is trivial if V is finite, for then we can obtain the desired conclusion by simply taking $W = V$.

The setting of Zorn's lemma is that of a finding of a maximal element in poset. Let's review the definitions: Let P be a poset (partially ordered set). Then a *chain* C is a totally ordered subset of P, i.e., every two elements of C are comparable. An *upper bound* for a set $S \subset P$ is an element $x \in P$ (upper bound) such that $x \geq y$ for each $y \in x$. The upper bound does not have to belong to S. An element $x \in P$ is said to be *maximal* if there is no $y \in P$ that is bigger than y.

Every finite poset has a maximal element (exercise!), but not all infinite posets do. For example, $\mathbb{N}$ with the usual ordering does not have a maximal element. The set $\mathbb{N} \cup \{\omega\}$ with the rule $x < \omega$ for every $x \in \mathbb{N}$ has a maximal element, ω.

*These notes are from `http://www.borisbukh.org/DiscreteMath12/notes_compactness.pdf`.

[1] Interested reader can find exposition via Tychonoff's theorem in the book "Ramsey theory" by Graham, Rothschild, Spencer

Theorem 2 (Zorn's lemma)**.** *If P is a non-empty poset, and every chain in P admits an upper bound, then P contains a maximal element.*

Proof that Zorn's lemma implies Compactness principle. A *partial r-coloring* of V is a r-coloring which may leave some vertices of V uncolored. Formally, it is just a map $\chi\colon W \to [r]$ where $W \subseteq V$. Pedantically, it is a pair (W, χ) consisting of a set $W \subseteq V$ and a map $\chi\colon W \to [r]$.

For purposes of the present proof we call a partial r-coloring (W, χ) *admissible* if for every finite $W' \subseteq V$ there is a proper coloring $\chi'\colon W' \to [r]$ such that $\chi'|_{W \cap W'} = \chi|_{W \cap W'}$. We express this situation by saying that the coloring χ' is *compatible* with χ. Note that an admissible coloring is necessarily proper on the set that it colors.

Let P be the poset consisting of all the admissible colorings, with the order given by $(W_1, \chi_1) \leq_P (W_2, \chi_2)$ if $W_1 \subseteq W_2$ and $\chi_2|_{W_1} = \chi_1$. We will apply Zorn's lemma to P, which is non-empty by the assumption of Compactness principle.

First, every chain in P has an upper bound: if C is a chain, then we can define a partial coloring χ by setting $\chi(x) = \chi'(x)$ if there is a partial coloring $(W, \chi') \in C$ such that $\chi'(x)$ is defined, and leaving x uncolored if no partial coloring that colors x exists. The χ is well-defined because C is a chain. It is proper because if $e \in E$ and χ colors all vertices of e, then since e is finite, we necessarily have $e \subset W$ for some $(W, \chi') \in C$.

Second, a maximal element of P is a coloring of all of V. Indeed, suppose (W, χ) is maximal in P, and $v \in V \setminus W$. For each color $c \in [r]$ let χ_c be the coloring obtained by extending χ to a coloring of $W \cup \{v\}$ by setting $\chi_c(v) = c$. Note that χ_c need not be admissible, or even proper. Since (W, χ) is maximal, for each $c \in [r]$ there is a finite set $W_c \subset V$ which admits no r-coloring that is compatible with χ_c. Let $\bar{W} = \bigcup_{c \in [r]} W_c$. Then $\bar{W}$ is finite, and so there is a proper r-coloring $\bar{\chi}\colon \bar{W} \to [r]$ that is compatible with χ (since (W, χ) is admissible). Let $\bar{c} = \bar{\chi}(v)$. We have reached a contradiction between previously established fact that $W_{\bar{c}}$ admits no coloring compatible with $\chi_{\bar{c}}$ and the fact that $\bar{\chi}$ is such a coloring of an even bigger set $\bar{W}$.

The coloring that corresponds to the maximal element P is a desired proper coloring of the whole hypergraph H, because for each $e \in E$ the finite graph induced on e is properly colored. □

Bonus material. Here is a sketch of a deduction of Zorn's lemma from Axiom of Choice. Pick any $x \in P$. For each ordinal α define x_α by the following rule: $x_0 = x$. If α is a successor ordinal, $\alpha = \beta + 1$, then set x_α to be any element larger than x (axiom of choice!). If α is a limit ordinal, then $C = \{x_\beta : \beta < \alpha\}$ is a chain in P. By the assumption of Zorn's lemma there is a upper bound y for C. Set $x_\alpha = y$.

The constructed sequence is strictly monotone, i.e., $x_\alpha < x_\beta$. Since there are more ordinals than elements of P, we have reached a contradiction.