Walk through Combinatorics: homework #2*
Due 3 October 2012

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let $s, t \in \mathbb{N}$. The vertex set of a graph G is a disjoint union of infinitely many blocks, each block being a set of size t. Inside any set of s distinct blocks there is an edge that goes between two different blocks. Show that in G there is an infinite path visiting no block more than once.

2. [2] Show that for each k and r there is a positive real number $c(k,r)$ such that every r-coloring of $[n]$ contains at least $\bigl(c(k,r) - o(1)\bigr)n^2$ monochromatic k-APs.

3. [2+1+2] For a coloring $\chi\colon [n] \to [r]$ a set $X \subset [n]$ is *rainbow* if all the elements of X receive different colors.

   (a) Show that for each k there is a $c = c(k) > 0$ such that if $[n]$ is colored in any number of colors, and no color occurs more than cn times, then there is a rainbow arithmetic progression of length k.

   (b) Deduce, using Szemerédi's theorem, that for each k there an n such that if $[n]$ is colored (in any number of colors), then there is a non-trivial arithmetic progression of length k that is either monochromatic or rainbow.

*This homework is from `http://www.borisbukh.org/DiscreteMath12/hw2.pdf`.

(c) Do part (b) without Szemerédi's theorem. [Hint: Define an auxillary coloring $\chi'\colon [n']^2 \to [r]^k$ by the rule $\chi'(x,y) = \big(\chi(x), \chi(x+y), \dots, \chi(x+(k-1)y)\big)$ and apply Gallai's theorem.]

4. [2+(2 extra credit)] The *step* of an arithmetic progression $\{a, a+d, a+2d, \dots, a+(k-1)d\}$ is defined to be $|d|$, the Euclidean norm of d.

   (a) Show that there is an r such that for each n there is a coloring $\chi\colon \mathbb{R}^n \to [r]$ that contains no monochromatic 3-AP with step 1. [Hint: choose $\chi(x)$ that depends only on $|x|$.]

   (b) [Extra credit] Show that for every r there is an n such that for every r-coloring χ of $\mathbb{R}^n$ there is a 4-AP $\{x_1, x_2, x_3, x_4\}$ with step 1 that satisfies $\chi(x_1) = \chi(x_4)$ and $\chi(x_2) = \chi(x_3)$. Here it is understood that x_1, x_2, x_3, x_4 are in order, i.e., they satisfy $2x_2 = x_1 + x_3$ and $2x_3 = x_2 + x_4$. [Hint: Consider the coloring of $\{0, \lambda, 2\lambda, 3\lambda\}^n \subset \mathbb{R}^n$, for a suitable λ.]