# Walk through Combinatorics: homework #1*
# Due 17 September 2012

Collaboration and use of external sources are permitted, but discouraged, and must be fully acknowledged and cited. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order.

1. [2] Let $k, l, m$ be arbitrary natural numbers. Let $ES(k, l, m)$ be the length of the longest sequence of real numbers that contains *neither*

   - a strictly increasing subsequence of length $k$, *nor*
   - a strictly decreasing subsequence of length $l$, *nor*
   - a constant subsequence of length $m$.

   Find $ES(k, l, m)$.

2. [2] Let $k > r$ be natural numbers. Consider an underdetermined system of homogeneous linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1k}x_k &= 0,\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2k}x_k &= 0,\\
\vdots \qquad\qquad & \quad \vdots\\
a_{r1}x_1 + a_{r2}x_2 + \cdots + a_{rk}x_k &= 0
\end{aligned}
$$

   with coefficients $a_{ij}$ that are integers and satisfy $|a_{ij}| \leq N$ for all $i$ and $j$. Show that there is a solution such that all the unknowns $x_1, \ldots, x_k$

*This homework is from `http://www.borisbukh.org/DiscreteMath12/hw1.pdf`.

are integers that are not all zero and that satisfy $|x_j| \le C_{k,r} N^{r/(k-r)}$. Here $C_{k,r}$ is a constant that depends only on $k$ and $r$. You do not have to find the smallest $C_{k,r}$ for which this holds.

3. [1] Recall that a set $P$ is in convex position if no point $p \in P$ is in convex hull of $P \setminus \{p\}$.

   Suppose that a finite set $P \subset \mathbb{R}^2$ contains no three collinear points, and that every four points of $P$ are in convex position. Show that $P$ is in convex position. [Hint: Consider the smallest subset of $P$ that is not in convex position.]

4. [Problem removed from the homework]

5. [2+1] Let $T$ be a tree having $k$ vertices (a *tree* is a connected graph containing no cycles)

   (a) If $n > (k-1)(l-1)$ and $K_n$ is colored red/blue, then $K_n$ contains a red $T$ or a blue $K_l$.

   (b) Show that this is not true if $n = (k-1)(l-1)$.

6. [2] Let $r \in \mathbb{N}$ be any fixed integer. Use probabilistic method to give a lower bound on $R_3(\overbrace{k, \ldots, k}^{r})$. Your answer should be an order-of-magnitude asymptotics for a fixed $r$ and large $k$ (i.e., is your bound exponential, doubly-exponential, or of an intermediate growth rate?).

7. [Extra credit, 2+2] Let $X$ be a set. A *$n$-letter word over alphabet $X$* is simply an element of $X^n$, i.e., a sequence of $n$ elements from $X$. Consider a word $w \in \big([k] \cup \{*\}\big)^n$. Let $C(w)$ be the set of all the words in $[k]^n$ that can be obtained by replacing stars by elements of $[k]$. For example, if $k = 2$ then

$$C(1*21*) = \{11211, 11212, 12211, 12212\}.$$

   In general, if $w$ has $m$ stars, then $|C(w)| = 2^m$.

   (a) Use Ramsey's theorem (or anything else) to show that there is a function $n_0(r, m)$ such that if $n \ge n_0(r, m)$ and $\chi\colon [2]^n \to [r]$ is a coloring of $n$-letter words over 2-letter alphabet, then there is a word $w \in ([2] \cup \{*\})^n$ with $m$ stars such that for $w' \in C(w)$ the color $\chi(w')$ depends only on the number of occurences of 1's and 2's in $w'$.

(b) Show that the analogous statement for 3-letter alphabet is false. Namely, show that there is an $r$ and $m$ such that for every $n$ there is a coloring $\chi\colon [3]^n \to [r]$ so that for no $w \in ([3] \cup \{*\})^n$ with $m$ stars the color of a word in $C(w)$ depends only on the number of occurences of 1's, 2's and 3's.