Exploring Combinatorics: homework #8*
Due 3 May 2013, at start of class

Collaboration is permitted, but is advised against, and all collaborators must be acknowledged. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in square brackets. The problems appear in no special order. All answers must be justified.

Practice problems (not graded; do not turn in)

1. For a permutation π of $[n]$ let $T(\pi)$ be the number of comparisons that QuickSort performs on input $\pi(1), \pi(2), \dots, \pi(n)$. Pick a permutation π at uniformly random from all permutations of $[n]$. Let $A_n = \mathbb{E}[T]$. Note that A_n is a number that depends on n.

   (a) Prove that
   $$A_n = n - 1 + \tfrac{1}{n}\sum_{k=1}^{n}(A_{k-1} + A_{n-k}).$$

   (b) Use generating functions to find the closed form for A_n. (Problem 1 from Homework #3 will come handy).

2. Suppose graph G contains $\alpha\binom{n}{3}$ triangles. Show that G must contain at least $\frac{1}{3}\alpha\binom{n}{2}$ edges. Can you prove a stronger bound?

3. Show that there is a set $A \subset [n]$ of size $\Omega(n^{1/5})$ that contains no six distinct elements $a_1, \dots, a_6$ satisfying $a_1 + a_2 + 2a_3 = a_4 + a_5 + 2a_6$.

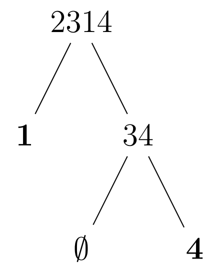
Homework problems (graded; turn in)

1. [2] Suppose that edges of K_n are colored into two colors, and assume that there are $\frac{1}{2}\binom{n}{2}$ edges of each color. Show that there are $\Omega(n^4)$ cliques on 4 vertices that *are not monochromatic*.

*This homework is from `http://www.borisbukh.org/CombinatoricsSpring13/hw8.pdf`.

2. [2] Think of $[n]^2$ as the n-by-n grid in the plane. Show that there is a set $A \subset [n]^2$ of size $|A| = \Omega(n^{10/7})$ that does not contain eight points that are vertices of two squares of equal area.

3. [2] Consider the recursive QuickSort algorithm as presented in class, and in the book. Suppose QuickSort is called with input π. Let $D(\pi)$ be the number of calls to QuickSort with input of size 1. For example, if $\pi = 2314$, then $D(\pi) = 2$ as seen from the execution tree on the right (the calls counted by D are in bold).



   Suppose π is chosen uniformly at random. Find the closed form for $\mathbb{E}[D]$.

4. [2] Suppose α is a real number satisfying $0 \leq \alpha < 1/2$. Prove that there is a number β (depending on α) satisfying $\beta > 1/4$ such that the following holds. For every edge-coloring of the complete graph K_n into two colors with $\alpha\binom{n}{2}$ edges colored red and $(1-\alpha)\binom{n}{2}$ edges colored blue, there are at least $\beta\binom{n}{3} + O(n^2)$ monochromatic triangles.