Exploring Combinatorics: homework #7*
Due 22 April 2013, at start of class

Collaboration is permitted, but is advised against, and all collaborators must be acknowledged. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in square brackets. The problems appear in no special order. All answers must be justified.

Practice problems (not graded; do not turn in)

1. A graph is *almost-bipartite* if it can be made bipartite by removing a single edge. Show that a random graph is almost surely not almost-bipartite.

2. A tournament is k-*crazy* if for every two disjoint sets L and W of k players each, there is a player p who lost to every player in L, and won against every player in W. Show that for every natural number k there is a k-crazy tournament with at least $2k$ players.

3. A *Hamilton cycle* in an n-player tournament is a permutation of players $p_1, p_2, \ldots, p_n$ such that the player p_{i+1} beats the player p_i for all $i \in [n-1]$ and p_1 beats p_n. Show that there is an n-player tournament with at least $(n-1)!/2^n$ Hamilton cycles.

4. Pick a number m uniformly at random from $[n]$. Show the expected number of distinct prime factors of m is
$$\Theta(\log \log n).$$

5. If π is a permutation of $[n]$, and $i \in [n]$, we call the number i *bouncy* if $\pi(\pi(i)) = i$, but $\pi(i) \neq i$. Compute the expected number of bouncy numbers in a random permutation.

*This homework is from `http://www.borisbukh.org/CombinatoricsSpring13/hw7.pdf`.

Homework problems (graded; turn in)

1. [1] Suppose X is a random variable on a finite probability space (Ω, P). Show that if $\mathbb{E}[X] = 1$, then $\mathbb{E}[X^2] \geq 1$.

2. [2] Pick a random permutation π uniformly among all permutations of $[n]$. Treating π as a string of n numbers, let X be the length of the longest increasing subsequence of π. Prove that $\mathbb{E}[X] \geq \frac{1}{2}\sqrt{n}$. (*Extra credit*: a valid proof of the bound $\mathbb{E}[X] \geq c\sqrt{n}$ is worth $2c - 1$ extra credit points).

3. (a) [1] Let (Ω, P) be a finite probability space. Suppose $A_1, \dots, A_n$ is a set of n independent events such that $0 < \Pr(A_i) < 1$ for all $i = 1, \dots, n$. Show from the definition of independent random events that $|\Omega| \geq 2^n$. (Hint: show that replacing some events by their complements produces a new set of independent events.)

   (b) [1] For each value of n, give an example of a finite probability space, and $n + 1$ events in this probability space such that every subset of n of these events is independent, but the set of all the $n + 1$ events is not. (Partial credit is awarded for solution to the case $n = 3$.)

4. [2] In a graph G, a *dominating set* is a set S of vertices, such that each vertex of G is either in S or is adjacent to a vertex of S. Suppose G has n vertices and every two vertices are joined by a path of at most two edges. Show that G contains a dominating set of size $O(\sqrt{n \log n})$. (*Hint*: consider the neighbors of a single vertex.)

5. [2] Find the maximum number of edges that a Hasse diagram of a poset on n elements can have.