Exploring Combinatorics: homework #6*
Due 5 April 2013, at start of class

Collaboration is permitted, but is advised against, and all collaborators must be acknowledged. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in square brackets. The problems appear in no special order. All answers must be justified.

Practice problems (not graded; do not turn in)

1. [0] Explain why the lower bound on Ramsey numbers from Theorem 11.3.1 is a special case of Theorem 10.1.4 on 2-coloring of set systems.

2. [0+0]

   (a) Given a Boolean function f on n variables, show that there is a formula of length $O(n2^n)$ that expresses f.

   (b) Given a Boolean function f on n variables, show that there is a formula of length $O(2^n)$ that expresses f.

3. [0] Prove that if (Ω, P) is a finite probability space, and $A_1, \dots, A_n \subset \Omega$ then
$$P\Big(\bigcup_{i=1}^{n} A_i\Big) = \sum_{\substack{I \subset [n] \\ I \neq \emptyset}} (-1)^{|I|+1} P(A_I),$$
where $A_I = \bigcap_{i \in I} A_i$.

4. [0] Suppose $n \geq 4$ and $\mathcal{F}$ is a collection of n-element subsets of some ground set X. Prove that if
$$|\mathcal{F}| < \frac{4^{n-1}}{3^n}$$
then there is a coloring of X with 4 colors such that every color appears at least once in every set in $\mathcal{F}$.

*This homework is from `http://www.borisbukh.org/CombinatoricsSpring13/hw6.pdf`.

Homework problems (graded; turn in)

1. [1] Use Ramsey's theorem to prove that for every k there is an n such that every sequence of n distinct real numbers contains a monotone subsequence of k real numbers. (You *must* use Ramsey's theorem, and *cannot* use the Erdős–Szekeres theorem.)

2. [1+1] An *edge-3-coloring* of a complete graph K_n is an assignment to each edge of K_n one of the three colors: red, green or blue. A *monochromatic* k-*clique* is a set of k vertices such that every edge connecting them is of same color.

   (a) Show that if $n \geq 3^{3k}$, then every edge-3-coloring of K_n contains a monochromatic k-clique.

   (b) Show that if $k \geq 3$, and $n < \sqrt{3}^{k-1}$, then there is a way to color edges of K_n into 3 colors so that there is no monochromatic k-clique.

3. [2] Show that if there is a real number p such that $0 \leq p \leq 1$ and
$$\binom{n}{s} p^{\binom{s}{2}} + \binom{n}{t} (1-p)^{\binom{t}{2}} < 1,$$
then the Ramsey number $r(s,t)$ satisfies $r(s,t) > n$.

4. [1+1] Let k and n be natural numbers. Let $x_1, \ldots, x_n$ be Boolean variables. A k-*disjunction* is a Boolean formula of the form $l_1 \vee l_2 \vee \cdots \vee l_k$, where each l_i is equal to one of $x_1, \ldots, x_n$ or one of $\neg x_1, \ldots, \neg x_n$, and any x_i occurs at most once. For example, $x_1 \vee \neg x_5 \vee x_2$ is a 3-disjunction, but neither $x_1 \vee \neg x_5 \vee \neg x_1$ nor $x_1 \vee x_6$ are. A k-*CNF formula with* m *clauses* is a formula of the form $\phi_1 \wedge \cdots \wedge \phi_m$, where each ϕ_i is a k-disjunction. For example, $(x_1 \vee \neg x_5 \vee x_2) \wedge (x_3 \vee x_1 \vee \neg x_2)$ is a 3-CNF formula with 2 clauses.

   (a) Show that for every k-CNF formula with *strictly fewer* than 2^k clauses there is an assignment of truth values to the variables such that the formula evaluates to "true".

   (b) Show that for every k there is a k-CNF formula with *exactly* 2^k clauses that always evaluates to "false".

5. [2] Show that for every natural number n there is a set $S \subset [n]$ such that $|S| = \Omega(n^{1/4})$ and for no four distinct elements s_1, s_2, s_3, s_4 of S we have $s_1 + s_2 = s_3 + s_4$. (Choose S at random.)