Exploring Combinatorics: homework #5*
Due 22 March 2013, at start of class

Collaboration is permitted, but is advised against, and all collaborators must be acknowledged. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in square brackets. The problems appear in no special order. All answers must be justified.

Practice problems (not graded; do not turn in)

1. [0] Suppose G is a bipartite graph with n vertices in each of the two parts of the bipartition. Let $K_{2,t}$ denote the graph on $t+2$ vertices $\{v_1, v_2, u_1, \ldots, u_t\}$, in which $v_i u_j$ is an edge for all choices of i and j. Show that if G contains no $K_{2,t}$, then G has at most $O\big(\sqrt{t}n^{3/2}\big)$ edges.

2. (a) [0] Give an example of a family of 2^{n-1} subsets of $[n]$ such that every two sets in the family intersect.

   (b) [0] Show that there is no family of more than 2^{n-1} subsets of $[n]$ such that every two sets in the family intersect.

3. [0] For each natural number n, give an example of a sequence of length n^2 of distinct real numbers that contains no monotone subsequence of length $n+1$.

4. [0] Consider two sequences $(a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ of distinct real numbers. Show that that indices $i_1, \ldots, i_k$, $1 \le i_1 < \cdots < i_k \le n$, always exist with $k = \lceil n^{1/4} \rceil$ such that the subsequence determined by them in both a and b are monotone (all 4 combinations are allowed, e.g, “increasing in a, decreasing in b”, “decreasing in both a and b, etc.).

*This homework is from `http://www.borisbukh.org/CombinatoricsSpring13/hw5.pdf`.

Homework problems (graded; turn in)

1. [1+1] Two permutations $\pi\colon [n] \to [n]$ and $\pi'\colon [n] \to [n]$ are said to *intersect* if there is an $i \in [n]$ such that $\pi(i) = \pi'(i)$. For example, the permutations 24315 and 52413 intersect, but 42153 and 13524 do not.

   (a) For each $n \geq 1$, give an example of a family of permutations of $[n]$ that contains $(n-1)!$ permutations, and such that every two permutations intersect.

   (b) Prove that there is no family of pairwise intersecting permutations that contains more than $(n-1)!$ permutations.

2. [2] Suppose π_1, π_2, π_3 are three permutations of $[n]$. Think of them as three shuffled sequences of numbers from 1 through n. Show that some pair of these sequences contains a common subsequence of length at least $\Omega(n^{1/3})$. (*Hint*: there are two possible solutions. One of them invokes Erdős–Szekeres theorem twice.)

3. [1+1+(extra credit)1] Considers strings over n-letter alphabet. Such a string is called *universal* if it contains every set of the letters as a substring[1]. For example, the 13-character-long string

   abcdeabdacebd

   contains all $2^5 = 32$ subsets of the 5-letter alphabet {a,b,c,d,e}, e.g., the set {b,d,e} occurs as a substring "ebd".

   (a) Show that there is a universal string of length $O(2^n\sqrt{n})$. (*Hint*: The key to the solution is in one of the proofs of Sperner's theorem in the book.)

   (b) Show that there is no universal string shorter than $\binom{n}{\lfloor n/2 \rfloor}$ characters.

   (c) Show that there is a universal string of length $O(2^n)$.

4. [2] The Erdős–Szekeres bound on Ramsey numbers tells us that $r(3,4) \leq \binom{3+4-2}{3-1} = 10$. Prove that in fact $r(3,4) \leq 9$.

[1] A *substring* is a contiguous sequence of letters in a string. For example, "dab" is a substring of "abracadabra", but "brad" is not.