Exploring Combinatorics: homework #4*
Due 1 March 2013, at start of class

Collaboration is permitted, but is advised against, and all collaborators must be acknowledged. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in square brackets. The problems appear in no special order. All answers must be justified.

Practice problems (not graded; do not turn in)

1. [0] Let $D(n)$ be the number of permutations π of the set $[n]$ such that $\pi(i) \neq i$ for all i (see section 3.8). Show that $|D(n) - n!/e| < 1/(n+1)$. Deduce that $D(n) = \lfloor n!/e + 1/2 \rfloor$ for all $n \geq 1$.

2. [0] Find a sequence $a_0, a_1, a_2, \ldots$ that satisfies $\sum_{k=0}^{n} a_k a_{n-k} = \binom{n+1}{1}$. Is the sequence unique?

3. [0] Explain what goes wrong if one tries to derive a formula for an area of a planar triangle in a similar way to the derivation of the formula for the area of a spherical triangle.

4. [0] Let n be a natural number. Find a closed form for $\binom{n-2/3}{2n}$. (Answer in terms of factorials, powers and/or binomial coefficients whose both lower and upper indices are integral.)

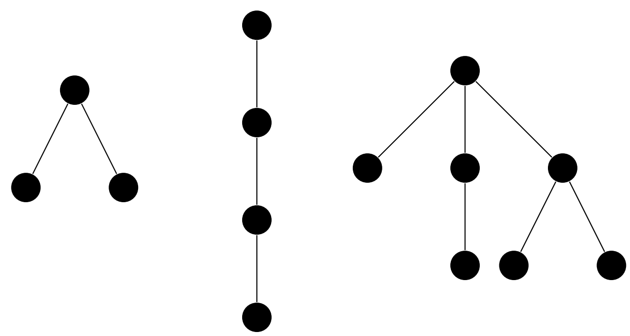
5. [0] Let n be a large integer. What is the probability of three randomly chosen numbers from $[n]$ having no common factor?

6. [0] How many ways are there to place 30 red and blue chairs around a large round table so that the resulting arrangement is not periodic? (An arrangement is periodic if there is a rotation which transforms it into itself.)

*This homework is from `http://www.borisbukh.org/CombinatoricsSpring13/hw4.pdf`.

Homework problems (graded; turn in)

1. [1+1+(1 extra credit)] An *ordered tree* is defined recursively: If $k \geq 0$ is an integer, then any tree that consists of a root vertex and an ordered k-tuple $(T_1, \dots, T_k)$ of subtrees, where each T_i is itself an ordered tree, is an ordered tree. (Careful! This recursive definition is wicked, for it has no base case, or does it?). Three examples of ordered trees are drawn below:



   (a) Let s_n be the number of ordered trees on n vertices. Find the generating function $S(x) = \sum_{n \geq 0} s_n x^n$.

   (b) Use part (a) to find the closed-form formula for s_n.

   (c) Find a bijective proof of the formula for s_n from part (b).

2. [2] Let $p_1, p_2, \dots, p_k$ be all the prime numbers not exceeding $\frac{1}{2} \log_2 n$. Use the inclusion–exclusion principle to show that among natural numbers 1 through n there are $(1 - 1/p_1)(1 - 1/p_2) \cdots (1 - 1/p_k)n + O(\sqrt{n})$ numbers that are not divisible by any of the primes $p_1, p_2, \dots, p_k$.

3. [2] How many ways are there to seat n married couples at a round table with $2n$ chairs in such a way that the couples never sit next to each other? Two seating plans differing by a rotation are considered different. (*Hint:* Use the inclusion–exclusion principle with A_i being the set of seat plans where i'th couple seats together. The answer is a single sum.)

4. [1+1] Let $n > 0$ be an even integer. Suppose $\mathcal{F} \subset 2^{[n]}$ is a family of sets that contains no three distinct sets A, B, C that satisfy $A \subset B \subset C$.

   (a) Show that $|\mathcal{F}| \leq 2\binom{n}{n/2}$.

   (b) Show that $|\mathcal{F}| \leq \binom{n}{n/2} + \binom{n}{n/2+1}$.

5. [2] There are $2n$ distinct points marked on a circle. We want to divide them into pairs and connect the points in each pair by a segment (chord) in such a way that these segments do not intersect. Show that the number ways to do so is the n'th Catalan number C_n.