Exploring Combinatorics: homework #3*
Due 15 February 2013, at start of class

Collaboration is permitted, but is advised against, and all collaborators must be acknowledged. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order. All answers must be justified.

1. [1+1+1] Let $H_n = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$ be the n'th harmonic number.

   (a) Show that the generating function for H_n is
   $$\sum_{n \geq 0} H_n x^n = -\frac{\ln(1-x)}{1-x}.$$
   Note that $H_0 = 0$ by convention for the empty sums.

   (b) Compute the generating function for nH_n.

   (c) Use (a) and (b) to find the closed form for $\sum_{0 \leq k < n} H_k$.

2. [2] Prove the following upper bound on $n!$
   $$n! \leq e\sqrt{n}(n/e)^n.$$
   Use the second proof of $n! \leq en(n/e)^n$ from the book as a starting point, and subtract areas of corresponding triangles.

3. [2] Show that the number $(6 + \sqrt{37})^{999}$ has at least 999 zeros following the decimal point. (Solutions using high-precision computer calculations will not be accepted.)

4. [1+1]

   (a) Let $m \geq 1$ be a fixed integer. Show that the product of primes p such that $m < p \leq 2m$ is at most $\binom{2m}{m}$.

*This homework is from `http://www.borisbukh.org/CombinatoricsSpring13/hw3.pdf`.

(b) Use (a) to prove that $\pi(n) = O(n/\log n)$, where $\pi(n)$ is the number of primes less than or equal to n. (Hint: first bound the number of primes between m and $2m$.)

5. [2] Let $p_k(n)$ be the number of partitions of n into k parts, i.e., the number of solutions to

$$n = m_1 + \cdots + m_k$$

in natural numbers $m_1, \ldots, m_k$ such that the order of the summands does not matter. Show that for each integer $t \geq 0$ there is an integer $f(t)$ such that all elements of the sequence $p_1(t+1), p_2(t+2), p_3(t+3), \ldots$ are equal to $f(t)$ from some point on.