Exploring Combinatorics: homework #2*
Due 1 February 2013, at start of class

Collaboration is permitted, but is advised against, and all collaborators must be acknowledged. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order. All answers must be justified.

1. [2] Give a combinatorial proof of

$$\left\{ {n+1 \atop m+1} \right\} = \sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop m} \right\}.$$

2. [1+1] Prove the following for all integers $n \geq 0$, and all real numbers x:

   (a) $x^n = \sum_k \left\{ {n \atop k} \right\} x(x-1)(x-2)\cdots(x-k+1)$,

   (b) $x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x(x+1)(x+2)\cdots(x+k-1)$.

3. [2] Let the sequence s_i be defined by $s_0 = 1$, $s_1 = 0$, $s_2 = -5$, $s_3 = 0$ and $s_n = 2s_{n-1} - s_{n-2}$ for $n \geq 4$. Find the generating function $S(x) = \sum_{i \geq 0} s_i x^i$.

4. [2] For all integers $n, r \geq 0$ compute

$$\sum_{k=0}^{r} (-1)^k \binom{n}{k} \binom{n}{r-k}.$$

5. [1+1]

   (a) Give a combinatorial proof of

$$\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

   (b) Use (a) to compute $\sum_{i=1}^{n} i^2$. (*Hint*: Consider $k = 2$)

*This homework is from `http://www.borisbukh.org/CombinatoricsSpring13/hw2.pdf`.