Exploring Combinatorics: homework #1*
Due 21 January 2013, at start of class

Collaboration is permitted, but is advised against, and all collaborators must be acknowledged. Collaboration may involve only discussion; all the writing must be done individually.

The number of points for each problem is specified in brackets. The problems appear in no special order. All answers must be justified.

1. [2] How many 9-digit numbers can be made of digits $1, 2, \ldots, 9$ so that for no value of i the i'th digit is i, and the number is not a palindrome. (A palindrome is a word, or a number that reads the same in either direction. So "rotator" and "121" are palindromes, but "abracadabra" and "42" are not.)

2. [2] Let $a, b \geq 1$ be integers. Give a combinatorial proof of

$$\binom{2a}{2b} \equiv \binom{a}{b} \pmod{2}.$$

3. [2] How many integral solutions of

$$x_1 + x_2 + x_3 + x_4 = 40$$

   satisfy $x_1 \geq 2$, $x_2 \geq 4$, $x_3 \geq -3$, and $x_4 \geq 0$. Your answer may involve binomial coefficients and factorials, but no summation signs, ellipses or a similar notation.

4. [2] For each integer $n \geq 0$ and each $m \in \{0, 1, 2\}$ let $T_m(n)$ be the number of subsets of an n-element set whose size leaves remainder m when divided by 3.

   Show that $T_m(n)$ is always equal to either $\lfloor 2^n/3 \rfloor$ or $\lceil 2^n/3 \rceil$. (*Hint:* Consider the differences $T_m(n) - 2^n/3$ for $m = 0, 1, 2$ and use induction).

*This homework is from `http://www.borisbukh.org/CombinatoricsSpring13/hw1.pdf`.