Algebraic Structures: homework #9*
Due 3 April 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let R and S be commutative rings with 1 and let P be a prime ideal in R. Are the following true or false? Justify.
   (a) For every ring homomorphism $\pi\colon R \to S$, the set $\pi(P)$ is a prime ideal in S.
   (b) For every ring homomorphism $\pi\colon S \to R$, the set $\pi^{-1}(P)$ is a prime ideal in S.
2. Let D be an integer that is not a square. Recall that the norm in $\mathbb{Z}[\sqrt{D}]$ is $N(a + b\sqrt{D}) = a^2 - Db^2$.
   (a) Prove that $r \in \mathbb{Z}[\sqrt{D}]$ is a unit if and only if $N(r) \in \{\pm 1\}$.
   (b) Prove that there are infinitely many units in $\mathbb{Z}[\sqrt{2}]$. [Find three units, the others are easier.]
3. Prove that $\mathbb{Z}[\sqrt{-2}]$ is a UFD.
4. Prove that if $r \in \mathbb{Z}$ can be written in the form $r = a^2 + b^2$ for some $a, b \in \mathbb{Q}$, then r can also be written as $r = c^2 + d^2$ for some $c, d \in \mathbb{Z}$.
5. Let F be a field.
   (a) Let $f \in F[x]$ and $\alpha \in F$. Show that $f(\alpha) = 0$ holds if and only if $x - \alpha \mid f$.
   (b) Let $\alpha \in F$. Show that $x - \alpha$ is a prime element in the ring $F[x]$.
   (c) Use the preceding parts to show that a non-zero polynomial $f \in F[x]$ of degree d has at most d roots.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw9.pdf`.