

Algebraic Structures: homework #8*

Due 27 March 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots$ are ideals in a ring R . Show that $\bigcup_{j=1}^{\infty} I_j$ is also an ideal in R .
2. Show that every ideal in the ring of matrices $M_2(\mathbb{R})$ is principal. [Note that this does not mean that the ring is a PID since $M_2(\mathbb{R})$ is noncommutative.]
3. Give an example of a group ring RG , with $G \neq 1$, which contains no zero divisors. Justify your example. [Hint: note that the group G must be infinite, for RG to have no zero divisors.]
4. Let R be a commutative ring, and I is an ideal in R . Show that the set

$$\{r \in R : \exists n \in \mathbb{Z}_+ r^n \in I\}$$

is an ideal in R .

5. Prove that the ring $\mathbb{Q}(\sqrt{2})$ contains no subring isomorphic to $\mathbb{Q}(\sqrt{3})$.

*This homework is from <http://www.borisbukh.org/AlgebraicStructures23/hw8.pdf>.