Algebraic Structures: homework #7*
Due 20 March 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let p, q, r be distinct prime numbers. Show that a group G of order pqr contains a normal subgroup of order either p, q, or r. [Sylow]

2. Recall that $\mathrm{Aut}(H)$ is the group of automorphisms of a group H. Denote by $\phi\colon \mathrm{Aut}(H) \to \mathrm{Aut}(H)$ the identity homomorphism. Define the group $\mathrm{Hol}(H) \stackrel{\text{def}}{=} H \rtimes_\phi \mathrm{Aut}(H)$. Show that $(h, k) \cdot h_0 \stackrel{\text{def}}{=} hk(h_0)$ defines an action of the group $\mathrm{Hol}(H)$ on H.

3. Let R be a commutative ring with 1. Let $x \in R$ be an element satisfying $x^n = 0$ for some n (such an x is called *nilpotent*).

   (a) Show that $1 + x$ is a unit. [Hint: try small n first, and then generalize.]

   (b) Show that $u + x$ is a unit for every unit $u \in R$.

4. Let R be a non-trivial ring with 1, and $M_n(R)$ be the ring of n-by-n matrices with entries in R. Show that the center of $M_n(R)$ contains only the diagonal matrices. [Hint: consider matrices almost all of whose entries are 0.]

5. The letter $\mathbb{Q}$ denotes the ring of rational numbers with the usual operations.

   (a) Prove that $\mathbb{Q}$ contains infinitely many distinct subrings. [Hint: a subring is also an additive subgroup.]

   (b) Prove that $\mathbb{Q}$ contains uncountably many distinct subrings.

   [Part (b) clearly implies (a), but starting with (a) is going to be easier.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw7.pdf`.