Algebraic Structures: homework #6*
Due 27 February 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. [The statement is wrong. Problem is removed from the homework.]
   Let H be a p-group. Show that the following two statements are equivalent:
   (a) The group H is cyclic.
   (b) The group H has exactly one subgroup of order p.
2. Let G be a finite abelian group.
   (a) Let p be a prime dividing $|G|$. Prove that G contains a subgroup of order p. [Possible approach: Use induction on $|G|$. Consider quotient G/N for some cyclic subgroup N of G.]
   (b) Prove that G contains a subgroup of every order dividing $|G|$. [Use part (a) and induction.]
3. Show that if P, P' are distinct Sylow p-subgroups of G, then there are $x \in P$ and $y \in P'$ that do not commute.
4. Suppose that A, B, C are normal subgroups of a group G. Prove that $ABC = \{abc : a \in A, b \in B, c \in C\}$ is also a normal subgroup of G.
5. Let H be any subgroup of a group G. Define an infinite sequence of subgroups as follows $H_1 = H$, and $H_{i+1} = N_G(H_i)$. Show that if G is finite, then there is an n such that $H_n = H_{n+1} = H_{n+2} = H_{n+3} = \cdots$.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw6.pdf`.