# Algebraic Structures: homework #5*
# Due 20 February 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Consider $A_n$ as a subgroup of the symmetric group on $\mathbb{Z}_+ \stackrel{\text{def}}{=} \{1, 2, 3, \cdots\}$ in the natural way, i.e., the elements of $A_n$ permute $\{1, 2, 3, \ldots, n\}$ and leave the other elements fixed. Let $A_\infty \stackrel{\text{def}}{=} \bigcup_{n \in \mathbb{Z}_+} A_n$.
   (a) Prove that $A_\infty$ is subgroup of $S_{\mathbb{Z}_+}$.
   (b) Prove that $A_\infty$ is simple.
2. (a) Prove that every element of $S_n$ is a product of at most $n-1$ transpositions.
   (b) Prove that every element of $S_n$ is a product of at most $n/2$ transpositions and 3-cycles.
3. Suppose $H$ is a subgroup of a group $G$. Show that $K = \bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of $G$ and that it is the largest normal subgroup contained in $H$ (i.e., every normal subgroup of $G$ contained in $H$ is contained in $K$).
4. Let $G$ be a finite group, and consider the action of $G$ on itself by left multiplication. Let $\pi \colon G \to S_G$ be the corresponding homomorphism. Show that if $x \in G$ is an element of order $n$ and $|G| = mn$, then $\pi(x)$ is a product of $m$ $n$-cycles.
5. Generalize any one problem from homeworks #1 through #4. You must say which problem you are generalizing, state your generalization, and provide a solution to that generalization. [ Saying that "$X$ is a generalization of $Y$" means that $X$ implies $Y$. You do not need to prove that your generalization is indeed a generalization, but it must be a strict generalization, i.e., $X = Y$ is not allowed. ]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw5.pdf`.