Algebraic Structures: homework #4*
Due 13 February 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Suppose K and L are normal subgroups of a finite group G. Let H be the group generated by the set $K \cup L$. Prove that $H \trianglelefteq G$. [The assumption that G is finite is not necessary, but we did not discuss what "generated by" means if the generating set is infinite.]

2. For which integers $n \geq 3$ does there exist a normal subgroup of D_{2n} not all of whose elements are rotations? Justify.

3. Let G be a group, let X be the set of all of its subgroups. For $g \in G$ and $H \in X$, let $g \cdot H \stackrel{\text{def}}{=} gHg^{-1}$. Show that this defines an action of G on X.

4. Let $G = \mathbb{R}/\mathbb{Z}$.

   (a) Prove that $\phi\colon G \to G$ defined by $x \mapsto 2x = x + x$ is a surjective homomorphism.

   (b) Deduce the existence of a normal subgroup N of G such that $1 \neq N \neq G$ and $G/N \cong G$. What is N?

5. Let H, K be subgroups of a group G.

   (a) Show that $H \cap K \leq G$.

   (b) Assume that $|G : H|$ and $|G : K|$ are finite.
   Show that $|G : H \cap K| \leq |G : H||G : K|$.

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw4.pdf`.