Algebraic Structures: homework #12*
Due 24 April 2023, at 9am

Collaboration and use of external sources are permitted, but must be fully acknowledged and cited. You will get most out of the problems if you tackle them on your own. Collaboration may involve only discussion; all the writing must be done individually.

1. Let k be a positive integer. Prove that there exists a polynomial $P_k \in \mathbb{Z}[x]$ such that
$$\cos kx = P_k\big(\cos x\big).$$
[Hint: guess a similar formula for $\sin kx$ and prove both by induction.]

2. True or false: Suppose points $(0,0)$ and $(1,0)$ are given on the plane. Then, using only straightedge and compass, we can construct any point of the form (a,b) as long as both a,b lie in some field extension $E/\mathbb{Q}$ satisfying $[E:\mathbb{Q}] = 2^n$ for some $n = 0,1,2,\dots$. Justify.

3. Let $f \in \mathbb{Z}[x]$ be a polynomial of degree 3. Show that there is a $r \in \mathbb{Z}$ such that $f + r$ is irreducible.

4. Let $f \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 1$ with $f(0) \neq 0$.
   (a) Show that $x^n f(1/x) \in \mathbb{Q}[x]$.
   (b) Show that f is irreducible if and only if $x^n f(1/x)$ is irreducible.

5. Generalize any one problem from homeworks #5 through #11. You must say which problem you are generalizing, state your generalization, and provide a solution to that generalization. [Saying that "X is a generalization of Y" means that X implies Y. You do not need to prove that your generalization is indeed a generalization, but it must be a strict generalization, i.e., $X = Y$ is not allowed.]

*This homework is from `http://www.borisbukh.org/AlgebraicStructures23/hw12.pdf`.